CONTACT GEOMETRY IN THE RESTRICTED THREE-BODY PROBLEM
MINI-COURSE LECTURE NOTES

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ABSTRACT. These are expanded notes for an online mini-course taught for postgraduate students at UDELAR, Montevideo, Uruguay, in November 2020, remotely from the Mittag-Leffler Institute in Djursholm, Sweden.

CONTENTS

1. Introduction.

The current set of notes is an attempt to put into context a series of very recent (and yet unpublished) results of the author in co-authorship with Otto van Koert [MvK, MvK], and the spin-off [M20]. They also serve the purpose of introducing and threading together a collection of basic and important notions, disseminated across the literature, with the main driving motivation coming from a very old and famous problem; namely, the three-body problem. We shall be, therefore, mainly interested in Hamiltonian dynamics, and the intended audience is that with a dynamical backround/interest; a good deal of openness towards topological/geometric/holomorphic techniques is also recommended. We make no assumptions on previous knowledge on contact or symplectic techniques; but we move at a fast pace.

We shall start from the basics of contact and symplectic geometry, the geometries of classical mechanics, and move on to the more topological notion of open book decompositions in the context of contact topology and Giroux's correspondence. We will then make a dynamical jump to discuss the notion of global hypersurfaces of section and adapted dynamics, discussing examples along the way. After paving the road, we focus on the three-body problem (more precisely, a simplified version, the circular restricted case=CR3BP) with the main intereset being the *spatial* problem where the small mass is allowed to move anywhere, as opposed to the *planar* problem, which historically has been of central interest. We sketch some historical account of Poincaré's original approach in the planar problem, discuss classical fixed-point theorems and perturbative results, as well as non-perturbative results coming from holomorphic curve theory à-la Hofer-Wysocki-Zehnder [HWZ98]. We then introduce the main results of [MvK, MvK, M20], which include:

- Existence of adapted open book decompositions for the spatial CR3BP in the low-energy range (Thm. M);
- Existence of Hamiltonian return maps reducing the dynamics to dimension 4 (Thm. N);
- A generalization of the classical Poincaré-Birkhoff theorem for Liouville domains in arbitrary even dimensions (Thm. O);
- The construction by the author of the *holomorphic shadow*, which associates to the SCR3BP (whenever the planar dynamics is convex, and energy is low) a Reeb dynamics on S^3 which is adapted to a trivial open book (Thm. R); and (perturbative) dynamical applications.

We remark that the first two results are valid for arbitrary mass-ratio and are therefore non-perturbative. We also point out that the second result, while a general fixed-point theorem, hasn't so far seen an application to the SCR3BP, for which the generalized notion of a twist condition introduced in [MvK] seems, as of yet, perhaps unsuitable. The third result, while of theoretical interest, might perhaps lead to insights on the original problem coming from 3-dimensional dynamics; this is work in progress. In fact, everything in the last sections should be considered work in progress. So the reader is advised to proceed accordingly, and perhaps get excited enough to contribute to this growing body of work.

Needless to say, this account will be very biased towards the author's interests; the subject is too vast to make it proper justice. The experienced reader is encouraged to complain to the author for misinterpretations, misrepresentations, omissions, or mistakes. Disseminated across the text we leave a series of digressions, intended for non-experts and newcomers, which the reader might choose to skip without affecting the understanding of the main body. They take up a significant part of the document, in the hope to illustrate the richness of the material. We also sprinkle exercises here and there, mostly well-known facts and not very technically demanding, although oftentimes relevant for the main discourse.

2. Basic concepts.

We start with the basic concepts underlying the general principles of classical mechanics.

2.1. **Symplectic geometry.** Roughly speaking, symplectic geometry is the geometry of phase-space (where one keeps track of position and velocities of classical particles, and so it is a theory in even dimensions). Formally, a *symplectic manifold* is a pair (M,ω), where M is a smooth manifold with $\dim(M)=2n$ even, and $\omega \in \Omega^2(M)$ is a two-form (the *symplectic form*) satisfying:

- (closedness) $d\omega = 0$;
- (non-degeneracy) $\omega^n = \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is nowhere-vanishing, and hence a volume form. Equivalently, the map
$$\mathfrak{X}(M) \to \Omega^1(M)$$
$$X \mapsto i_X\omega = \omega(X,\cdot)$$
is a linear isomorphism.

Exercise 1. Prove that the above two definitions of non-degeneracy for ω are equivalent.

Note that symplectic manifolds are always orientable. We assume that M is always oriented by the orientation induced by the symplectic form.

Example 2.1. (From classical mechanics).

- (Phase-space) $(\mathbb{R}^{2n},\omega_{std})$, where, writing $(q,p)\in\mathbb{R}^{2n}=\mathbb{R}^n\oplus\mathbb{R}^n$ (q =position, p =momenta), we have
$$\omega_{std} = -d\lambda_{std} = dq\wedge dp,$$
where $\lambda_{std}=pdq$ is the standard *Liouville form*. Here we use the shorthand notation $dq\wedge dp = \sum_{i=1}^n dq_i\wedge dp_i$ and similarly $pdq=\sum_{i=1}^n p_i dq_i$.
- (cotangent bundles) (T^*Q,ω_{std}), where Q is a closed n-manifold, and ω_{std} is defined invariantly as
$$\omega_{std} = -d\lambda_{std},$$
with
$$(\lambda_{std})_{(q,p)}(\eta) = p(d_{(q,p)}\pi(\eta)),$$
also called the standard Liouville form. Here, q is a point in the base, and p a covector in T_qQ^*, and
$$\pi : T^*Q \to Q$$
is the natural projection to the base. Note that phase-space corresponds to the case $Q=\mathbb{R}^{2n}$.

A general important feature of symplectic manifolds (or, more like, the reason for their existence) is that they are locally modelled on phase-space:

Theorem A (Darboux's theorem for symplectic manifolds)**.** *If $p\in(M,\omega)$ is an arbitrary point in a symplectic manifold, we can find local charts centered at p, so that (M,ω) is isomorphic to standard phase-space $(\mathbb{R}^{2n},\omega_{std})$ in this local chart.*

The notion of isomorphism we use above is the obvious one: two symplectic manifolds (M_1,ω_1) and (M_2,ω_2) are *symplectomorphic* if there exists a diffeomorphism $f : M_1 \to M_2$ satisfying $f^*\omega_2 = \omega_1$. Darboux's theorem is usually interpreted as saying that, unlike in Riemannian geometry where the curvature is a local isometry invariant, there are no local invariants for symplectic manifolds (they locally all look the same).

Hamiltonian dynamics. From a dynamical perspective, symplectic manifolds are the natural geometric space where one can study Hamiltonian dynamics, via the *Hamiltonian formalism*. On a cotangent bundle T^*Q, the idea is to model the motion of a particle moving along the manifold Q, subject to the principle of minimization of energy/action associated to a given physical problem.

In general, we start with a symplectic manifold (M,ω), and a *Hamiltonian* $H : M \to \mathbb{R}$, which is simply a function (which we assume C^1, say), thought of as the *energy* function of the mechanical system. The symplectic form implicitly defines a vector field $X_H \in \mathfrak{X}(M)$ (the *Hamiltonian vector field* or *Hamiltonian gradient* of H) via the equation

$$i_{X_H}\omega = dH.$$

Note that this uniquely defines X_H due to non-degeneracy of ω. The above equation is the global, invariant version for the following:

Exercise 2. (Fundamental example: Hamilton's equation) Check that whenever $(M,\omega) = (\mathbb{R}^{2n},\omega_{std})$, we have

$$X_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right) = \frac{\partial H}{\partial p}\partial_q - \frac{\partial H}{\partial q}\partial_p.$$

In other words, a solution $x(t) = (q(t),p(t))$ to the ODE $\dot{x}(t) = X_H(x(t))$ is precisely a solution to the Hamilton equations

$$\begin{cases} \dot{q} = & \frac{\partial H}{\partial p} \\ \dot{p} = & -\frac{\partial H}{\partial q} \end{cases}$$

By Darboux's theorem, we see that, locally, solutions to the Hamiltonian flow are solutions to the above.

Exercise 3. (Simple harmonic oscillator) Solve the Hamilton equations for the simple harmonic oscillator $H : \mathbb{R}^2 \to \mathbb{R}$, $H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$, where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency, k is the spring constant, m is the mass of a classical particle with position x and momenta p. Draw the Hamiltonian orbits (i.e. the phase-space diagram).

Remark 2.2. The Hamiltonian usually also depends on time. We have assumed for simplicity that it does not, i.e. it is autonomous. We will see that this will hold for the simplified versions of the three body problem we will consider.

In the above symplectic formalism, it is a fairly straightforward matter to write down the fundamental conservation of energy principle (in the autonomous case):

Theorem B. *(Conservation of energy) Assume H is autonomous. Then*

$$dH(X_H) = 0.$$

In other words, the level sets $H^{-1}(c)$ are invariant under the Hamiltonian flow.

This is also usually written down using the *Poisson bracket* as

$$\{H,H\} = 0,$$

which is another way of saying that H is preserved under the Hamiltonian flow of itself, or that H is a conserved quantity (or integral) of the motion. The proof fits in one line:

$$dH(X_H) = i_{X_H}\omega(X_H) = \omega(X_H,X_H) = 0,$$

since ω is skew-symmetric.

2.2. **Contact geometry.** Contact geometry is, roughly speaking, the odd-dimensional analogue of symplectic geometry, and arises on level sets of Hamiltonians satisfying a suitable convexity assumption (see Prop. 2.5). Formally, a *(strict) contact manifold* is a pair (X, α), where X is a smooth manifold with $\dim(X) = 2n-1$ odd, and $\alpha \in \Omega^1(X)$ is a 1-form (the *contact form*) satisfying the *contact* condition:

$$\alpha \wedge d\alpha^{n-1} \neq 0 \text{ is nowhere-vanishing, and hence a volume form.}$$

Contact manifolds are therefore orientable (see Remark 2.4 below). The codimension-1 distribution $\xi = \ker \alpha \subset TM$ (a choice of hyperplane at each tangent space, varying smoothly with the point), is called the *contact structure* or *contact distribution*, and (M, ξ) is a *contact manifold*.

Example 2.3.

- (standard) The standard contact form on $\mathbb{R}^{2n-1} = \mathbb{R} \oplus \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1} \ni (z, q, p)$ is

  $$\alpha_{std} = dz - pdq,$$

  where we again use the short-hand notation $pdq = \sum_{i=1}^n p_i dq_i$.

- (First-jet bundles) Given a manifold Q, its first-jet bundle $J^1(Q) \to Q$, by definition, has total space the collection of all possible first-derivatives of maps $f : Q \to \mathbb{R}$. The fiber over q is as all possible tuples $(q, f(q), d_q f)$, and so $J^1(Q) \cong \mathbb{R} \times T^*Q$. It carries the natural contact form

  $$\alpha = dz + \lambda_{std},$$

  where z is the coordinate on the first factor, and λ_{std} is the standard Liouville form on T^*Q; note that the standard contact form corresponds to the case $Q = \mathbb{R}^{n-1}$.

- (contactization) More generally: If $(M, \omega = d\lambda)$ is an exact symplectic manifold, then its *contactization* is

  $$(\mathbb{R} \times M, dz + \lambda),$$

  where z is the coordinate in the first factor.

The contact condition should be thought of as a *maximally non-integrability* condition, as follows. Recall the following theorem from differential geometry:

Theorem C (Frobenius' theorem)**.** *If $\alpha \wedge d\alpha \equiv 0$, then $\xi = \ker \alpha \subset TM$ is integrable. That is, there are codimension-1 submanifolds whose tangent space is ξ.*

The condition in Frobenius' theorem is equivalent to $d\alpha|_\xi \equiv 0$. The contact condition is the extreme opposite of the above: $d\alpha|_\xi > 0$ is symplectic, i.e. non-degenerate. In fact:

Exercise 4. If $Y \subset (X, \xi)$ is a submanifold of a $(2n-1)$-dimensional contact manifold so that $TY \subset \xi$ (i.e. Y is *isotropic*), then $\dim(Y) \leq n-1$. The isotropic submanifolds of maximal dimension $2n-1$ are called *Legendrians*.

The analogous theorem of Darboux in the contact category is the following:

Theorem D (Darboux's theorem for contact manifolds)**.** *If $p \in (X, \lambda)$ is an arbitrary point in a strict contact manifold, we can find a local chart $U \cong \mathbb{R}^{2n-1}$ centered at p, so that $\lambda|_U = \alpha_{std}$.*

Reeb dynamics. Whereas a contact manifold is a geometric object, a *strict* contact manifold is a dynamical one, as we shall see below. Note first that the choice of contact form for a contact structure ξ is not unique: if α is such a choice, then $\nu\alpha$ is also, for any smooth positive function $\nu > 0$. This is in fact the only ambiguity.

Given a contact form α, it defines an autonomous dynamical system on X, generated by the *Reeb vector field* $R_\alpha \in \mathfrak{X}(X)$. This is defined implicitly via:

- $i_{R_\alpha} d\alpha = 0$;
- $\alpha(R_\alpha) = 1$.

To understand the above, note that, since $d\alpha|_\xi$ is symplectic, the kernel of $d\alpha$ is the 1-dimensional distribution $TX/\xi \subset TX$. This is trivialized (as a real line bundle) via a choice of contact form, which also gives it an orientation induced from the one on M. The Reeb vector field then lies in this 1-dimensional distribution; the second condition normalizes it so that it points precisely in the positive direction with respect to the co-orientation. We emphasize that the Reeb vector field depends significantly on the contact form, and not the contact structure; different choices give, in general, very different dynamical systems.

Remark 2.4. There are also examples of contact manifolds which are not globally co-orientable (e.g. the space of contact elements); we will not be concerned with those.

The Reeb flow φ_t has the property that it preserves the geometry in a strict way, i.e. it is a *strict contactomorphism*. This means that $\varphi_t^*\alpha = \alpha$, or in other words, the Reeb vector field generates a (strict) local symmetry of the (strict) contact manifold. This fact easily follows from the Cartan formula:

$$\frac{d}{dt}\varphi_t^*\alpha = di_{R_\alpha}\alpha + i_{R_\alpha}d\alpha = d(1) + 0 = 0,$$

and so $\varphi_t^*\alpha = \varphi_0^*\alpha = \alpha$.

More generally, a (not necessarily strict) contactomorphism is a diffeomorphism f such that $f^*(\xi) = \xi$, or $f^*\alpha = \nu\alpha$ for some strictly positive smooth function ν.

The bridge. The fundamental relationship between symplectic and contact geometry lies in the following. If the symplectic form $\omega = d\lambda$ is exact (which can only happen if the symplectic manifold is open, by Stokes' theorem), then we have a *Liouville* vector field V, defined implicitly via

$$i_V\omega = \lambda,$$

where we again use non-degeneracy of ω. To understand this vector field, consider φ_t the flow of V. The Cartan formula implies

$$\frac{d}{dt}\varphi_t^*\omega = di_V\omega + i_Vd\omega = d\lambda = \omega,$$

and so, integrating, we get

$$\varphi_t^*\omega = e^t\omega.$$

Taking the top wedge power of this equation: $\varphi_t^*\omega^n = e^{nt}\omega^n$, and we see that the symplectic volume grows exponentially along the flow of V, i.e. φ_t is a *symplectic dilation*.

Assume that $X \subset (M, \omega = d\lambda)$ is a co-oriented codimension-1 submanifold, and the Liouville vector field is positively transverse to X. Then we obtain a volume form on X contraction:

$$0 < i_V\omega^n|_X = ni_V\omega \wedge \omega^{n-1}|_X = n\lambda \wedge d\lambda^{n-1}|_X = \alpha \wedge d\alpha^{n-1},$$

where $\alpha = \lambda|_X$. We have proved:

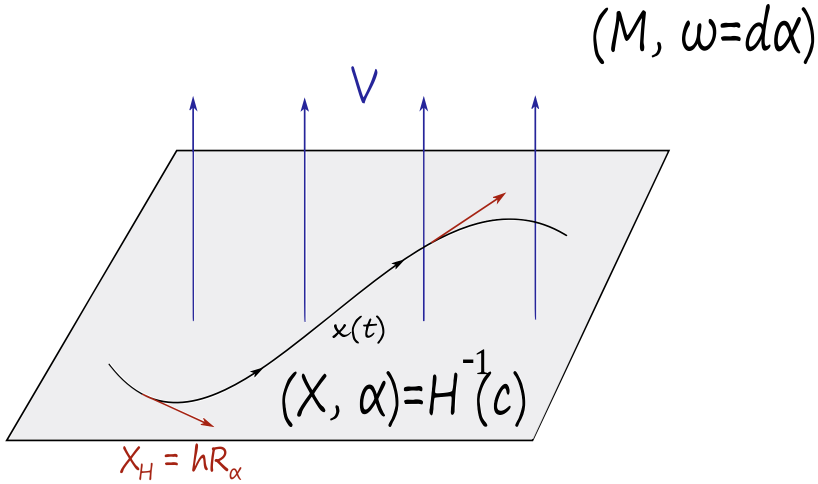

FIGURE 1. The fundamental relationship between contact and symplectic geometry is summarized here.

Proposition 2.5. *If $\omega = d\lambda$, and the associated Liouville vector field V is positively transverse to X, then $(X, \alpha = \lambda|_X = i_V\omega|_X)$ is a strict contact manifold.*

A hypersurface X as in the above proposition is then called *contact-type*. The most relevant example to keep in mind, is when $X = H^{-1}(c)$ is the level set of a Hamiltonian (in fact, locally this is always the case). In this situation:

Proposition 2.6. *If $X = H^{-1}(c)$ is contact-type, then the Reeb dynamics on X is a positive reparametrization of the Hamiltonian dynamics of H.*

Exercise 5. Prove this proposition (this is an easy but important exercise).

In other words, *Reeb dynamics on contact-type Hamiltonian level sets is dynamically equivalent to Hamiltonian dynamics.* See Figure 1 for an abstract sketch.

Example 2.7.

- (star-shaped domains) Assume that $X \subset \mathbb{R}^{2n}$ is *star-shaped*, i.e. it bounds a compact domain D containing the origin, and the radial vector field $V = q\partial_q + p\partial_p = \partial_r$ is positively transverse to X (with the boundary orientation). Since V is precisely the Liouville vector field associated to λ_{std}, every star-shaped domain is contact-type.
- (standard contact form on S^3) As a particular case, let $S^3 = \{z \in \mathbb{R}^4 : |z| = 1\} \subset \mathbb{R}^4$ be the round 3-sphere. Then $S^3 = H^{-1}(1/2)$, where $H : \mathbb{R}^4 \to \mathbb{R}$, $H(z) = \frac{1}{2}|z|^2$, and it is star-shaped. Writing $z = (z_1, z_2) = (x_1, y_1, x_2, y_2)$, the radial vector field
$$V = \frac{1}{2}r\partial_r = \frac{1}{2}(x_1\partial_{x_1} + y_1\partial_{y_1} + x_2\partial_{x_2} + y_2\partial_{y_2})$$
is Liouville and induces the contact form
$$\alpha = i_V\omega_{std}|_{S^3} = \lambda_{std}|_{S^3} = \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3}$$

on S^3 whose Reeb vector field is

$$R_\alpha = 2(x_1\partial_{y_1} - y_1\partial_{x_1} + x_2\partial_{y_2} - y_2\partial_{x_2}).$$

Its Reeb flow is, in complex coordinates, $\varphi_t(z_1, z_2) = e^{2\pi it}(z_1, z_2)$, whose orbits are precisely the fibers of the Hopf fibration $S^3 \ni (z_1, z_2) \mapsto [z_1 : z_2] \in \mathbb{C}P^1$. In particular, this flow is periodic, and all orbits have the same period.

As a side remark: the Hopf fibration $\pi : S^3 \to S^2 = \mathbb{C}P^1$ is an example of what is usually called a *prequantization bundle*, i.e. the contact form α is a connection form whose curvature form on the base is symplectic. In other words, $d\alpha = i\pi^*\omega_{FS}$ for a symplectic form ω_{FS} on S^2, and its Reeb orbits are the S^1-fibers (here, ω_{FS} is the Fubini-Study metric on $\mathbb{C}P^1$, and the line bundle associated to the principal S^1-bundle π is $\mathcal{O}(1) \to \mathbb{C}P^1$; see the digression on line bundles below).

- (ellipsoids) Given $a, b > 0$, define the *ellipsoid*

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\},$$

a star-shaped domain. The restriction of the symplectic form ω_{std} is a symplectic form on $E(a, b)$, and its boundary $\partial E(a, b)$ inherits a contact form $\lambda_{std}|_{\partial E(a,b)}$ whose Reeb flow is

$$\varphi_t(z_1, z_2) = (e^{2\pi at}z_1, e^{2\pi bt}z_2).$$

In particular, if a, b are rationally independent, then this Reeb flow has only two periodic orbits, passing through the points $z_1 = 0$, or $z_2 = 0$. If $a = b$, $E(a, a)$ is the unit ball, and we recover the Hopf flow along the standard $S^3 = \partial E(a, a)$.

- (Unit cotangent bundle and geodesic flows) Given a manifold Q, choose a Riemannian metric on TQ (which induces a metric on T^*Q), and consider its unit cotangent bundle

$$S^*Q = \{(q, p) \in T^*Q : |p| = 1\}.$$

We have $S^*Q = H^{-1}(1/2)$, where $H : T^*Q \to \mathbb{R}$, $H(q, p) = \frac{|p|^2}{2}$ is the kinetic energy Hamiltonian. The radial vector field $V = p\partial_p$ on each fiber is the Liouville vector field associated to λ_{std}, and is positively transverse to S^*Q. It follows that $\alpha_{std} := \lambda_{std}|_{S^*Q}$ is a contact form, and $(S^*Q, \xi_{std} = \ker \alpha_{std})$ is called the standard contact structure on S^*Q. Its Reeb dynamics is the (co)geodesic flow. We see that *a geodesic flow is a particular case of a Reeb flow.*

Exercise 6. (Hopf fibration) Check that the Reeb orbits of the standard contact form on S^3 are indeed the S^1-fibers $t \mapsto e^{it}(z_1, z_2)$ of the Hopf fibration $S^3 \ni (z_1, z_2) \mapsto [z_1 : z_2] \in \mathbb{C}P^1$, where we write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

Symplectization. Given a contact form α on X, its *symplectization* is the symplectic manifold

$$(\mathbb{R} \times X, \omega = d(e^t\alpha)).$$

The Liouville vector field is $V = \partial_t$, which is positively transverse to all slices $\{t\} \times X$, where it induces the contact form $i_V\omega = e^t\alpha$. Note that the Reeb dynamics is the same in each slice (i.e. it is only rescaled by a constant positive multiple). In fact, the symplectization is the "universal neighbourhood" for every contact-type hypersurface:

Proposition 2.8. *Let $X \subset (M, \omega)$ be a contact-type hypersurface, with $\omega = d\lambda$ exact near X. Then we can find sufficiently small $\epsilon > 0$, and an embedding*

$$\Phi : (-\epsilon, \epsilon) \times X \hookrightarrow M,$$

so that $\Phi^*\omega = d(e^t\alpha)$ where $\alpha = \lambda|_X$.

Exercise 7. Prove the proposition. *Hint: use the Liouville flow.*

In other words, contact manifolds are always contact-type in some symplectic manifolds, and vice-versa. We can summarize this discussion in the following motto: *contact geometry is $\mathbb{R}$-invariant symplectic geometry.*

Remark 2.9. One also calls the symplectic manifold $(\mathbb{R} \times X, \omega = d(r\alpha))$ the symplectization of α; this is related to the above by the obvious change of coordinates $r = e^t$. We shall use the two interchangeably. Note that $X = \{t = 0\} = \{r = 1\}$.

Digression: examples of symplectic manifolds from complex algebraic/Kähler geometry.

Example 2.10.

- (Projective varieties) Complex projective space $\mathbb{C}P^n$ admits a natural symplectic form, called the *Fubini-Study* form ω_{FS}, defined as follows. Let
$$K : \mathbb{C}^n \to \mathbb{R}$$
$$K(z) = \log\left(1 + \sum_{i=1}^n |z_i|^2\right).$$
In homogeonous coordinates $(\zeta_0 : \cdots : \zeta_n)$ for $\mathbb{C}P^n$, let $U_\alpha = \{(\zeta_0 : \cdots : \zeta_n) : \zeta_\alpha \neq 0\}$ and
$$\varphi_\alpha : U_\alpha \to \mathbb{C}^n,$$
$$\varphi_\alpha(\zeta_0 : \cdots : \zeta_n) = \left(\frac{\zeta_0}{\zeta_i}, \dots, \frac{\zeta_{i-1}}{\zeta_i}, \frac{\zeta_{i+1}}{\zeta_i}, \dots, \frac{\zeta_n}{\zeta_i}\right) = (z_1^\alpha, \dots, z_n^\alpha)$$
be the standard affine chart around $(0 : \cdots : 1 : \cdots : 0)$. Let $K_\alpha = K \circ \varphi_\alpha$, and define
$$\omega_\alpha = \sqrt{-1}\partial\overline{\partial}K_\alpha = \sum_{i,j=1}^n h_{ij}(z^\alpha) dz_i^\alpha \wedge d\overline{z}_j^\alpha.$$
Here, one computes
$$h_{ij}(z^\alpha) = \frac{\delta_{ij}\left(1 + \sum_{i=1}^n |z_i^\alpha|^2\right) - z_i^\alpha \overline{z}_j^\alpha}{\left(1 + \sum_{i=1}^n |z_i^\alpha|^2\right)^2}$$
One checks that on overlaps $U_\alpha \cap U_\beta$, we have $\omega_\alpha = \omega_\beta$, and so we get a well-defined global ω_{FS} so that $\omega_{FS}|_{U_\alpha} = \omega_\alpha$. The K_α are what is called a local Kähler potential (or plurisubharmonic function) for the Fubini-Study form. Every algebraic/analytic projective variety inherits a symplectic form via restriction of the ambient Fubini-study form.
- (Affine varieties: Stein manifolds) The standard complex affine space $\mathbb{C}^n$ carries the standard symplectic form via the identification $\mathbb{C}^n = \mathbb{R}^{2n}$, which in complex notation is
$$\omega_{std} = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\overline{z}_j =: \frac{\sqrt{-1}}{2} dz \wedge d\overline{z} = -d\lambda_{std}$$

with $\lambda_{std} = \frac{\sqrt{-1}}{4}(zd\overline{z} - \overline{z}dz)$. This admits the standard plurisubharmonic function

$$f_{std}(z) = |z|^2,$$

i.e. $\omega_{std} = \sqrt{-1}\partial\overline{\partial}f_{std}$. This function is exhausting (i.e. $\{z : f(z) \leq c\}$ is compact for every $c \in \mathbb{R}$), and is a Morse function (with a unique critical point at the origin).

By analogy as with the projective case, a Stein manifold X is a properly embedded complex submanifold of $\mathbb{C}^n$, endowed with the restriction of the standard symplectic form, the standard complex structure i, and the standard plurisubharmonic function. One may further assume (after a small perturbation) that f_{std} defines a Morse function on X.

The above examples (projective and affine) are all instances of Kähler manifolds, i.e. the symplectic form is suitably compatible with an integrable complex structure, and with a Riemannian metric. One way to obtain Stein manifolds from projective varieties is to remove a collection of generic hyperplane sections, i.e. the intersection of the variety with the zero sets of generic homogeneous polynomials of degree 1. A confusing point is that the Liouville form (i.e. the primitive of the resulting symplectic form), depends on the number of sections, as we illustrate as follows in the case of $\mathbb{C}P^n$ as the projective variety.

Continued digression: relationship with line bundles, connections and Chern-Weil theory. First, as a general fact, we recall that the Picard group of $\mathbb{C}P^n$ (i.e. the group of isomorphism classes of holomorphic line bundles, with tensor product as group operation) is isomorphic to $\mathbb{Z}$, each $k \in \mathbb{Z}$ corresponding to a line bundle $\mathcal{O}(k)$. For $k \geq 0$, the holomorphic sections of $\mathcal{O}(k)$ are precisely homogeneous polynomials of degree k on the homogeneous coordinates; $\mathcal{O}(k)$ has no holomorphic sections for $k < 0$, but admits meromorphic sections given by Laurent polynomials with poles of total order k. Moreover, the first Chern class of a line bundle is by definition the Poincaré dual of $Z(s)$, the zero set of a section s, generic in the sense that it is transverse to the zero section. The zero set of a generic polynomial of degree k is, by definition, a hypersurface of degree k. For very degenerate cases (i.e. when the polynomial factorizes into linear polynomials), this consists of a collection of hyperplanes, i.e. zero sets of linear polynomials as e.g. $H = \{\zeta_i = 0\}$, with total multiplicity k. One should think of $\mathbb{C}P^1$, where this zero set is simply a collection of points with total multiplicity k. This translates to the fact that first Chern class of $\mathcal{O}(k)$ is $c_1(\mathcal{O}(k)) = kh \in H^2(\mathbb{C}P^n, \mathbb{R})$, where h is the hyperplane class, the Poincaré dual to the homology class $[H] \in H_{2n-2}(\mathbb{C}P^n, \mathbb{R})$ of any hyperplane H, and a generator of the cohomology of $\mathbb{C}P^n$. On the other hand, Chern-Weil theory says that c_1 is represented by the curvature 2-form of a connection on $\mathcal{O}(k)$ (e.g. the Chern connection associated to the standard Hermitian metric). In practice, this means the following: for $k \geq 0$, take a holomorphic section $s_k \in \Gamma(\mathcal{O}(k))$, and consider $F_k = \sqrt{-1}\partial\overline{\partial}\log(|s_k|^2)$, which a $(1,1)$-form, defined on $X_k := \mathbb{C}P^n \backslash Z(s_k)$. We further have $F_k = -dd^{\mathbb{C}}\log|s_k|^2$, where $d^{\mathbb{C}}$ is defined via $d^{\mathbb{C}}\alpha(X) = d\alpha(iX)$, and so F_k is exact on X_k. Moreover, it is symplectic on X_k, which becomes a subset of $\mathbb{C}^n$ after choosing affine charts, and is in fact a Stein manifold, where the appropiate Liouville form for the symplectic form F_k is $\lambda_k = -d^{\mathbb{C}}\log|s_k|^2$. In other words, projective space is obtained from X_k by compactifying with a divisor $Z(s_k)$ "at infinity". Thinking of s_k as providing a local trivialization of $\mathcal{O}(k)$ over X_k, one checks that different choices of local trivializations give different F_k which glue together to a global $(1,1)$-form which is no longer exact, and actually its cohomology class is precisely $c_1(\mathcal{O}(k))$. Note that by construction, any standard chart U_α is of the form $\mathbb{C}P^n \backslash Z(s_1) \cong \mathbb{C}^n$, and $\omega_{FS}|_{U_\alpha} = F_1$, i.e. ω_{FS} is the curvature of the Chern connection on $\mathcal{O}(1)$ and hence Poincaré dual to h.

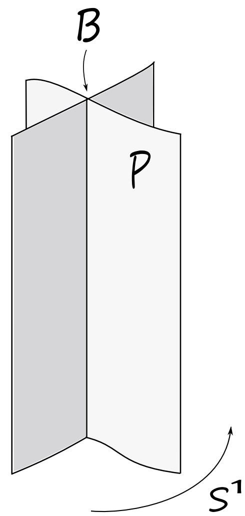

FIGURE 2. A neighbourhood of the binding look precisely like the pages of an open book, whose front page has been glued to its back page.

Exercise 8 (Further reading). If unfamiliar with the above material, make sense of the above discussion related to the examples coming from complex algebraic/Kähler geometry. Good references are Griffiths-Harris [GH], Huybrechts [Huy05], and many others.

2.3. **Open book decompositions.**

Definition 2.11. Let M be a closed manifold. A (concrete) open book decomposition on M is a fibration $\pi : M \backslash B \to S^1$, where $B \subset M$ is a closed, codimension-2 submanifold with trivial normal bundle. We further assume that $\pi(b, r, \theta) = \theta$ along some collar neighbourhood $B \times \mathbb{D}^2 \subset M$, where (r, θ) are polar coordinates on the disk factor.

Note that collar neighbourhoods of B exist, since they are trivializations of its normal bundle. B is called the *binding*, and the closure of the fibers $P_\theta = \overline{\pi^{-1}}(\theta)$ are called the *pages*, which satisfy $\partial P_\theta = B$ for every θ. We usually denote a concrete open book by the pair (π, B). See Figure 2.

The above concrete notion also admits an abstract version, as follows. Given the data of a typical page P (a manifold with boundary B), and a diffeomorphism $\varphi : P \to P$ with $\varphi|_B = id$, we can abstractly construct a manifold

$$M := \mathbf{OB}(P, \varphi) := B \times \mathbb{D}^2 \bigcup_\partial P_\varphi,$$

where $P_\varphi = P \times [0,1] \backslash (x, 0) \sim (\varphi(x), 1)$ is the associated mapping torus. By gluing the obvious fibration $P_\varphi \to S^1$ with the angular map $(b, r, \theta) \mapsto \theta$ defined on $B \times \mathbb{D}^2$, we see that this abstract notion recovers the concrete one. Reciprocally, every concrete open book can also be recast in abstract terms, where the choices are unique up to isotopy. However, while the two notions are equivalent from a topological perspective, it is important to make distinctions between the abstract and the concrete versions for instance when studying dynamical systems adapted to the open books (as we shall do below), since dynamics is in general very sensitive to isotopies.

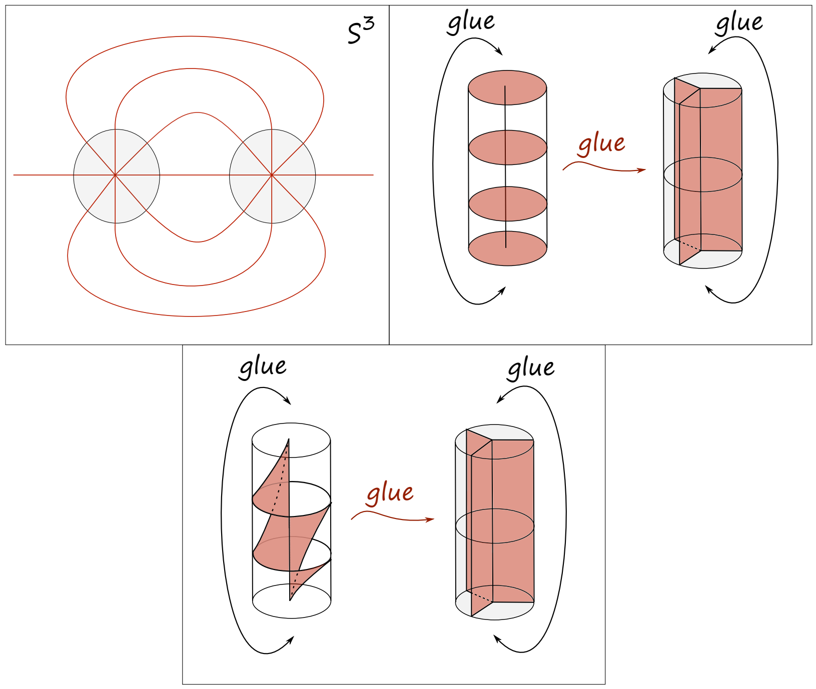

FIGURE 3. The disk-like pages of the trivial open book in S^3 (above) are obtained by gluing two foliations on two solid tori; similarly for its stabilized version (below), whose pages are annuli. Here we use the genus 1 Heegaard splitting for S^3.

Example 2.12.

- (trivial open book) Since the relative mapping class group of $\mathbb{D}^2$ is trivial, the only possible monodromy for an open book with disk-like pages is $S^3 = \mathbf{OB}(\mathbb{D}^2, \mathbb{1})$. Viewing $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, let $B = \{z_1 = 0\} \subset S^3$ be the binding (the unknot). The concrete version is e.g. $\pi : S^3 \backslash B \to S^1$, $\pi(z_1, z_2) = \frac{z_1}{|z_1|}$. See Figure 3.
- (stabilized version) We also have $S^3 = \mathbf{OB}(D^*S^1, \tau)$, where τ is the positive Dehn twist along the zero section S^1 of the annulus D^*S^1. A concrete version is $\pi : S^3 \backslash L \to S^1$, $\pi(z_1, z_2) = \frac{z_1 z_2}{|z_1 z_2|}$, where $L = \{z_1 z_2 = 0\}$ is the Hopf link. This is the positive stabilization of the trivial open book, an operation which does not change the manifold (see below). See Figure 3.
- (Milnor fibrations) More generally, let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial which vanishes at the origin, and has no singularity in S^3 except perhaps the origin. Then $\pi_f : S^3 \backslash B_f \to S^1$, $\pi_f(z_1, z_2) = \frac{f(z_1, z_2)}{|f(z_1, z_2)|}$, $B_f = \{f(z_1, z_2) = 0\} \cap S^3$, is an open book for S^3, called the Milnor fibration of the hypersurface singularity $(0, 0)$. The link B_f is the link of the singularity, and the binding of the open book, whereas the page is called the Milnor fibre. If f has no critical point at $(0, 0)$, then B_f is necessarily the unknot.

- We have $S^1 \times S^2 = \mathbf{OB}(D^*S^1, \mathbb{1})$. This can be easily seen by removing the north and south poles of S^2 (whose S^1-fibers become the binding), and projecting the resulting manifold $D^*S^1 \times S^1$ to the second factor.
- (Some lens spaces) We have $\mathbb{R}P^3 = \mathbf{OB}(D^*S^1, \tau^2)$, as follows from taking the quotient of the stabilized open book in S^3 via the double cover $S^3 \to \mathbb{R}P^3$. More generally, for $p \geq 1$, we have $L(p, p-1) = \mathbf{OB}(D^*S^1, \tau^p)$, and for $p \leq 0$, $L(p,1) = \mathbf{OB}(D^*S^1, \tau^p)$. Here, $L(p,q) = S^3/\mathbb{Z}_p$, is the lens space, where the generator $\zeta = e^{\frac{2\pi i}{p}} \in \mathbb{Z}_p$ acts via $\zeta \cdot (z_1, z_2) = (\zeta.z_1, \zeta^q.z_2)$. For $p = 0, 1, 2$, we recover the above examples.

Exercise 9. Prove the above claims: $L(p, p-1) = \mathbf{OB}(D^*S^1, \tau^p)$ for $p \geq 1$, $L(p,1) = \mathbf{OB}(D^*S^1, \tau^p)$, for $p \leq 0$. Try not to get confused with orientations (we have $L(p, p-1) = -L(p,1)$). The discussion below on Heegaard splittings helps.

In general, we have the following important result from smooth topology, which says that the open book construction achieves all closed, odd-dimensional manifolds:

Theorem E (Alexander ($\dim = 3$), Wilkelnkemper (simply-connected, $\dim \geq 7$), Lawson ($\dim \geq 7$), Quinn ($\dim \geq 5$))**.** *If M is closed and odd-dimensional, then M admits an open book decomposition.*

So far, we have discussed open books in terms of smooth topology. We now tie it with contact geometry, via the fundamental work of Emmanuel Giroux, which basically shows that contact manifolds can be studied from a purely topological perspective. One therefore usually speaks of the field contact *topology*, when the object of study is the contact manifold itself (as opposed e.g. to a Reeb dynamical system on the contact manifold).

If M is oriented and endowed with an open book decomposition, then the natural orientation on the circle induces an orientation on the pages, which in turn induce the boundary orientation on the binding. The fundamental notion is the following:

Definition 2.13 (Giroux)**.** Let (M, ξ) be an oriented contact manifold, and (π, B) an open book decomposition on M. Then ξ is *supported* by the open book if one can find a positive contact form α for ξ (called a *Giroux form*) such that:

(1) $\alpha_B := \alpha|_B$ is a positive contact form for B;
(2) $d\alpha|_P$ is a positive symplectic form on the interior of every page P.

Here, the a priori orientations on binding and pages are the ones described above.

The above conditions are equivalent to:

(1)′ $R_\alpha|_B$ is tangent to B;
(2)′ R_α is positively transverse to the interior of every page.

In the above situation, $(B, \xi_B = \ker \alpha_B)$ is a codimension-2 contact submanifold, i.e. $\xi_B = \xi|_B$.

Theorem F (Giroux)**.** *Every open book decomposition supports a unique isotopy class of contact structures. Any contact structure admits a supporting open book decomposition.*

Here, two contact structures are isotopic if they can be joined by a smooth path ξ_t of contact structures. An important result in contact geometry is *Gray's stability*, which says that isotopic contact structures are *contactomorphic*, i.e. there exists a diffeomorphism which carries one to the other. One may further assume that the pages in the above theorem are Stein manifolds, as discussed above. One may unequivocally use $\mathbf{OB}(P, \varphi)$ to denote the unique isotopy class of contact structures that this open book supports.

Giroux's result is actually much stronger in dimension 3, since it moreover states that the supporting open book is unique up to a suitable notion of *positive stabilization*, which can be thought of as two cancelling surgeries which therefore smoothly do not change the ambient manifold:

Theorem G (Giroux's correspondence)**.** *If* $\dim(M) = 3$*, there is a 1:1 correspondence*

$$\{\textit{contact structures}\}/\textit{isotopy} \longleftrightarrow \{\textit{open books}\}/\textit{pos. stabilization}$$

This bijection is why in dimension 3 one talks about Giroux's *correspondence*, which reduces the study of contact 3-manifolds to the topological study of open books. The analogous general uniqueness statement in higher-dimensions is an open question to this day. Let us emphasize that in the above result only the contact *structure* is fixed, and the contact form (and hence the dynamics) is auxiliary; Giroux's result is *not* dynamical, but rather topological/geometrical.

2.4. **Global hypersurfaces of section.** From a dynamical point of view, one wishes to adapt the underlying topology to the given dynamics, rather than vice-versa. We therefore make the following:

Definition 2.14. Given a flow $\varphi_t : M \to M$ of an autonomous vector field on an odd-dimensional closed manifold M carrying a concrete open book decomposition (π, B), we say that the open book is *adapted to the dynamics* if:

- B is φ_t-invariant;
- for each $x \in M \backslash B$ and P a page, then the orbit of x intersects the interior of P in the future, and in the past, i.e. there exists $\tau^+(x) > 0$ and $\tau^-(x) < 0$ such that $\varphi_{\tau^\pm}(x) \in \text{int}(P)$.

If φ_t is a Reeb flow, then the above is equivalent to asking that the (given) contact form is a Giroux form for the (auxiliary) open book. It follows from the definition, that each page is a *global hypersurface of section*, defined as follows:

Definition 2.15. (Global hypersurface of section) A *global hypersurface of section* for an autonomous flow φ_t on a manifold M is a codimension-1 submanifold $P \subset M$, whose boundary (if non-empty) is flow-invariant, and the orbit of every point in $M \backslash \partial P$ intersects the interior of P in the future and past.

Poincaré return map. Given a global hypersurface of section P for a flow φ_t, this induces a Poincaré return map, defined as

$$f : \text{int}(P) \to \text{int}(P),\ f(x) = \varphi_{\tau(x)}(x),$$

where $\tau(x) = \min\{t > 0 : \varphi_t(x) \in \text{int}(P)\}$. This is clearly a diffeomorphism. And, by construction, periodic points of f (i.e. points p for which $f^k(p) = p$ for some $k \geq 1$) are in 1:1 correspondence with closed *spatial* orbits (those which are not fully contained in the binding).

Moreover, in the case of a Reeb dynamics we have:

Proposition 2.16. *If* φ_t *is the Reeb flow of a contact form* α*, and* P *is a global hypersurface of section with induced return map* f*, then* $\omega = d\alpha|_P = d\lambda$*, with* $\lambda = \alpha|_P$*, is a symplectic form on* P*, and*

$$f : (P, \omega) \to (P, \omega)$$

is a symplectomorphism, i.e. $f^*\omega = \omega$*.*

In fact, f is an *exact* symplectomorphism, which means that $f^*\lambda = \lambda + d\tau$ for some smooth function τ (i.e. the return time). Differentiating this equation, we obtain $f^*\omega = \omega$. In dimension 2, a symplectic form is just an area form, and so the above proposition simply says that the return map is area-preserving.

The proof is quite simple: ω is symplectic precisely because the Reeb vector field (which spans the kernel of $d\alpha$) is transverse to P. For $x \in \text{int}(P)$, $v \in T_xP$, we have

$$d_xf(v) = d_x\tau(v)R_\alpha(f(x)) + d_x\varphi_{\tau(x)}(v).$$

Using that φ_t satisfies $\varphi_t^*\alpha = \alpha$, we obtain

$$\begin{aligned}(f^*\lambda)_x(v) &= \alpha_{f(x)}(d_xf(v))\\ &= d_x\tau(v) + (\varphi_{\tau(x)}^*\alpha)_x(v)\\ &= d_x\tau(v) + \lambda_x(v).\end{aligned} \tag{2.1}$$

Therefore

$$f^*\lambda = d\tau + \lambda, \tag{2.2}$$

which proves the proposition.

Remark 2.17. In general, the return map might not necessarily extend to the boundary, and indeed there are many examples on which this doesn't hold; this is a delicate issue which usually relies on analyzing the linearized flow equation along the normal direction to the boundary.

2.5. **Examples of adapted dynamics.** Let us discuss two important but simple examples of open books supporting a Reeb dynamics.

Hopf flow. The trivial open book on S^3, as well as its stabilized version, are both adapted to the Hopf flow.

Exercise 10. Prove this claim.

Exercise 11. More generally, prove that the trivial and stabilized open books on S^3 are adapted to the Reeb dynamics of every ellipsoid $E(a,b)$. Show that, in the trivial case, the return map on each page is the rotation by angle $2\pi\frac{a}{b}$; and in the stabilized case, we get a map of the annulus which rotates the two boundary components in the same direction (i.e. it is *not* a twist map). Interpret the dynamics in terms of the fixed points of these return maps (or lack thereof), and relate this phenomenon to the classical Brouwer fixed point theorem, and the Poincaré-Birkhoff theorem, as discussed below.

2.6. **Geodesic flow on S^n, and the geodesic open book.** We write

$$T^*S^n = \left\{(\xi,\eta) \in T^*\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} : \|\xi\| = 1,\ \langle \xi,\eta\rangle = 0\right\}.$$

The Hamiltonian for the geodesic flow is $Q = \frac{1}{2}\|\eta\|^2|_{T^*S^n}$ with Hamiltonian vector field

$$X_Q = \eta \cdot \partial_\xi - \xi \cdot \partial_\eta.$$

This is the Reeb vector field of the standard Liouville form λ_{std} on the energy hypersurface $\Sigma = Q^{-1}(\frac{1}{2}) = S^*S^n$. We have the invariant set

$$B := \{(\xi_0,\dots,\xi_n;\eta_0,\dots,\eta_n) \in \Sigma \mid \xi_n = \eta_n = 0\} = S^*S^{n-1}.$$

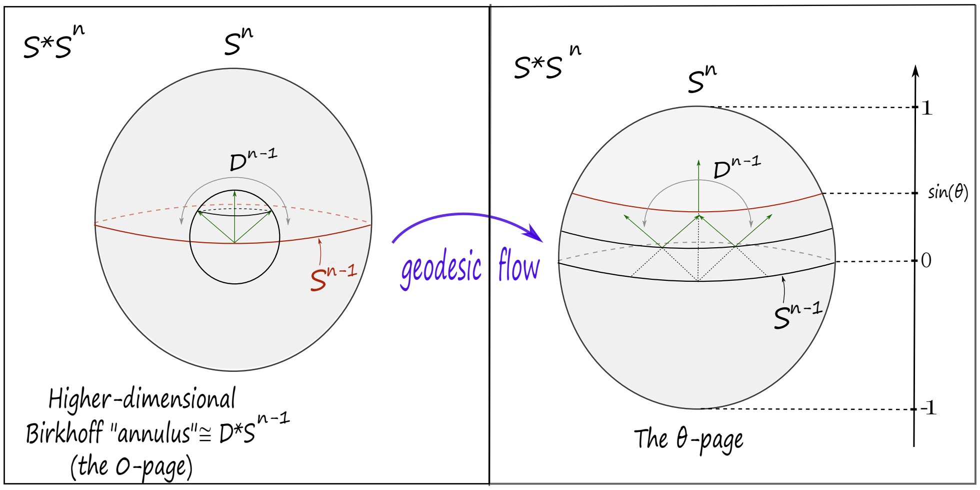

FIGURE 4. The geodesic open book for S^*S^n.

Define the circle-valued map

$$\pi_g : \Sigma \setminus B \longrightarrow S^1, \quad (\xi_0, \dots, \xi_n; \eta_0, \dots, \eta_n) \longmapsto \frac{\eta_n + i\xi_n}{\|\eta_n + i\xi_n\|}.$$

This is a concrete open book on S^*S^n, which we shall refer to as the *geodesic* open book. The page $\xi_n = 0$ and $\eta_n > 0$, i.e. the fiber over $1 \in S^1$, corresponds to a higher-dimensional version of the famous *Birkhoff annulus* (when $n = 2$), and is a copy of $\mathbb{D}^*S^{n-1}$. Indeed, it consists of those (co)-vectors whose basepoint lies in the equator, and which point upwards to the upper-hemisphere. See Figure 4.

We then consider the angular form

$$\omega_g = d\pi_g = \frac{\eta_n d\xi_n - \xi_n d\eta_n}{\xi_n^2 + \eta_n^2}.$$

We see that $\omega_g(X_Q) = 1 > 0$, away from B. This means that (B, π_g) is a supporting open book for Σ and the pages of π_g are global hypersurfaces of section for X_Q. In fact, all of its pages are obtained from the Birkhoff annulus by flowing with the geodesic flow. In terms of the contact structure $\xi_{std} = \ker \lambda_{std}$, this open book corresponds to the abstract open book $(S^*S^n, \xi_{std}) = \mathbf{OB}(\mathbb{D}^*S^{n-1}, \tau^2)$ supporting ξ_{std}. Here, $\tau : \mathbb{D}^*S^{n-1} \to \mathbb{D}^*S^{n-1}$ is an exact symplectomorphism defined by Arnold in dimension 4 in [A95] and extended by Seidel to higher-dimensions (see e.g. [Sei00]), and is a generalization of the classical Dehn twist on the annulus. For $n = 2$, we reobtain the open book $\mathbb{R}P^3 = S^*S^2 = \mathbf{OB}(\mathbb{D}^*S^1, \tau^2)$.

2.7. **Double cover of S^*S^2.** We focus on $n = 2$, and consider

$$S^*S^2 = \{(\xi, \eta) \in T^*\mathbb{R}^3 : \|\xi\| = 1, \langle \xi, \eta \rangle = 0\},$$

the unit cotangent bundle of S^2, with canonical projection $\pi_0 : S^*S^2 \to S^2$, $\pi_0(\xi, \eta) = \xi$. It is easy to see that the map

$$\Phi : S^*S^2 \to SO(3),$$

$$\Phi(\xi,\eta)=(\xi,\eta,\xi\times\eta),$$

is a diffeomorphism, where we view ξ,η as column vectors, and so $S^*S^2\cong SO(3)\cong\mathbb{R}P^3$. The projection π_0 on $SO(3)$ becomes $\pi_0(A)=A(e_1)$, i.e. the first column of the matrix $A\in SO(3)$. We have $\pi_1(S^*S^2)=\mathbb{Z}_2$, generated by the S^1-fiber. By definition, the double cover of $SO(3)$ is the Spin group Spin(3), which can be constructed as follows. Consider the quaternions

$$\mathbb{H}=\{a+bi+cj+dk : a,b,c,d\in\mathbb{R}\},$$

with $i^2=j^2=k^2=-1$, $ij=k, jk=i, ki=j$. We identify $S^3=Sp(1):=\{q\in\mathbb{H}:\|q\|=1\}$, and $\mathbb{R}^3=\mathrm{Im}(\mathbb{H})=\langle i,j,k\rangle$ the set of purely imaginary quaternions. The conjugate of $q=a+bi+cj+dk$ is $\overline{q}=a-bi-cj-dk$. We then define

$$p:S^3\to SO(3),$$

$$p(q)(v)=\overline{q}vq,$$

where $v\in\mathrm{Im}(\mathbb{H})=\mathbb{R}^3$. We have $\|\overline{q}vq\|=\|q\|^2\|v\|=\|v\|$, and $p(q)$ is seen to preserve orientation, so indeed $p(q)\in SO(3)$. Clearly $p(-q)=p(q)$, and the map p is in fact a double cover, so that $S^3=$ Spin(3).

Identifying i with e_1, we have $\pi_0(p(q))=p(q)(i)=\overline{q}iq$. A short computation gives

$$\overline{q}iq=(a+bi+cj+dk)^*i(a+bi+cj+dk)=(a^2+b^2-c^2-d^2)i+2(bc-ad)j+2(ac+bd)k.$$

On the other hand, the Hopf map may be defined as the map

$$\pi:S^3\to S^2,\ \pi(z_1,z_2)=(|z_1|^2-|z_2|^2,2\mathrm{Re}z_1\overline{z_2},2\mathrm{Im}z_1\overline{z_2}),$$

where we view $S^3=\{(z_1,z_2)\in\mathbb{C}^2;|z_1|^2+|z_2|^2=1\}$ and $S^2\subset\mathbb{R}^3$. Writing $q=a+bi+cj+dk=z_1+z_2j$, i.e. $z_1=a+ib$, $z_2=c+id$, one can easily check that

$$(|z_1|^2-|z_2|^2,2\mathrm{Re}z_1\overline{z_2},2\mathrm{Im}z_1\overline{z_2})=(a^2+b^2-c^2-d^2,2(bc-ad),2(ac+bd)).$$

We have proved the following:

Proposition 2.18. *The Hopf fibration is the fiber-wise double cover of the canonical projection π_0, i.e. we have a commutative diagram*

$$\begin{tikzcd}
S^1 \arrow[r, "z\mapsto z^2"] \arrow[d, hook] & S^1 \arrow[d, hook] \\
S^3=\mathit{Spin}(3) \arrow[r, "p"] \arrow[d, "\pi"] & SO(3)=S^*S^2 \arrow[d, "\pi_0"] \\
S^2 \arrow[r, equal] & S^2
\end{tikzcd}$$

2.8. **Magnetic flows and quaternionic symmetry.** On this section, we expose the beautiful construction of [AGZ] (to which we refer the reader for further details here omitted), relating the quaternions with Reeb flows on S^3, as double covers of magnetic flows on S^*S^2.

On S^2, consider an area form σ (the *magnetic field*), and the *twisted* symplectic form ω_σ, defined on T^*S^2 via

$$\omega_\sigma = \omega_{std} - \pi_0^*\sigma,$$

where $\pi_0 : T^*S^2 \to S^2$ is the natural projection. Fixing a metric g on S^2, the Hamiltonian flow of the kinetic Hamiltonian $H(q,p) = \frac{\|p\|^2}{2}$, computed with respect to ω_σ, is called the *magnetic flow* of (g,σ). Note that $\sigma = 0$ corresponds to the geodesic flow of g. Physically, the magnetic flow models the motion of a particle on S^2 subject to a magnetic field (the terminology comes from Maxwell's equations, which can be recast in this language). From now on, we fix σ to be the standard area form on S^2, with total area 4π, and g the standard metric with constant Gaussian curvature 1.

On S^*S^2, we can choose a connection 1-form α satisfying $d\alpha = \pi^*\sigma$, which is a contact form (usually called a *prequantization form*). We identify $T^*S^2\backslash S^2$ with $\mathbb{R}^+ \times S^*S^2$, and denoting by $r \in \mathbb{R}^+$ the radial coordinate, we have the associated symplectization form $d(r\alpha)$. Consider the S^1-family of symplectic forms

$$\omega_\theta = \cos\theta\, d(r\alpha) + \sin\theta\, d(r\alpha_{std}),\ \theta \in \mathbb{R}/2\pi\mathbb{Z},$$

defined on $\mathbb{R}^+ \times S^*S^2 = T^*S^2\backslash S^2$, where $d(r\alpha_{std}) = \omega_{std}$. The Hamiltonian flow of the kinetic Hamiltonian H, with respect to ω_θ, and along $r = 1$, is easily seen to be the magnetic flow of $(g, -\cot\theta \cdot \sigma)$ up to constant reparametrization. In particular, for $\theta = \pi/2 \bmod \pi$, we obtain the geodesic flow, whose orbits are great circles; for other values of θ the strength of the magnetic field increases, and the orbits become circles of smaller radius with an increasing left drift. For $\theta = 0 \bmod \pi$, the circles become points and the flow rotates the fibers of S^*S^2, i.e. this is the magnetic flow with "infinite" magnetic field.

We now construct the double covers of these magnetic flows on S^3, using the hyperkähler structure on $\mathbb{H} = \mathbb{R}^4 = \mathbb{C}^2$. We view S^3 as the unit sphere in $\mathbb{H}$. Every unit vector

$$c = c_1 i + c_2 j + c_2 k \in S^2 \subset \mathbb{R}^3$$

may be viewed as a complex structure on $\mathbb{H}$, i.e. $c^2 = -\mathbb{1}$. Denoting the radial coordinate on $\mathbb{R}^4$ by r, we obtain an S^2-family of contact forms on S^3 given by

$$\alpha_c = -2dr \circ c|_{TS^2},\ c \in S^2.$$

The Reeb vector field of α_c is $R_c = \frac{1}{2}c\partial_r$. Note that α_i is the standard contact form on S^3, whose Reeb orbits are the Hopf fibers.

We then consider the quaternionic action of S^3 on itself, given by

$$l_a : S^3 \to S^3$$

$$u \mapsto au,$$

for $a \in S^3$. Recall that we also have the action of S^3 on S^2 via the $SO(3)$-action of the previous section, i.e. $a \cdot c = p(a)(c) = ac\overline{a} \in S^2$, for $a \in S^3, c \in S^2$, and $p : S^3 \to SO(3)$ the Spin group double cover. One checks directly that $(l_a)_*\alpha_c = \alpha_{ac\overline{a}} = \alpha_{a\cdot c}$. In particular, $(l_a)_*\alpha_i = \alpha_{\pi(a)}$, where π is the Hopf fibration.

On the other hand, the stabilizer of $i \in S^2$ under the S^3-action is the circle

$$\text{Stab}(i) = \{\cos(\varphi) + i\sin(\varphi) : \varphi \in S^1\} \cong S^1 \subset S^3.$$

The action of an element in this subgroup on S^3 then fixes α_i, but reparametrizes its Reeb orbits, i.e. rotates the Hopf fibers. We then consider an S^1-subgroup $\{a_\theta\} \subset S^3$ of unit quaternions which are transverse to this stabilizer, intersecting it only at the identity, given by

$$a_\theta = \cos(\theta/2) + k\sin(\theta/2),\ \theta \in [0,\pi]$$

for which

$$\pi(a_\theta) = a_\theta i \overline{a}_\theta = i\cos\theta + j\sin\theta.$$

Define

$$\alpha_\theta := \alpha_{\pi(a_\theta)} = \cos\theta\ \alpha_i + \sin\theta\ \alpha_j,$$

with Reeb vector field $R_\theta := R_{\pi(a_\theta)}$. One further checks that

$$\alpha_\theta = p^*(\cos\theta\ \alpha + \sin\theta\ \alpha_{std}),$$

and so

$$\widetilde{\omega}_\theta := d\alpha_\theta = p^*\omega_\theta|_{S^*S^2}$$

is the double cover of the twisted symplectic form ω_θ along the unit cotangent bundle (alternatively, we can also think of $\widetilde{\omega}_\theta$ as being defined on $\mathbb{R}^4\backslash\{0\} = \mathbb{R}^+ \times S^3$ as the symplectization of α_θ). We have obtained:

Theorem H. *[AGZ] There are contact forms α_i, α_j and an S^1-action on S^3, sending α_i to contact forms $\alpha_\theta = \cos\theta\ \alpha_i + \sin\theta\ \alpha_j$, $\theta \in S^1$, such that the Reeb flow of α_θ doubly covers the magnetic flow of ω_θ.*

Remark 2.19. Note that for $\theta = 0$, corresponding to the infinite magnetic flow, this reduces to the statement of Prop. 2.18. For $\theta = \pi/2$, this says that we can lift the geodesic flow on S^2 to (a rotated version of) the Hopf flow. Of course, this statement depends on choices; we could have arranged that the lift is precisely the Hopf flow by changing our choice of coordinates.

2.9. **The magnetic open book decompositions.** We now tie the previous discussion with open book decompositions. We have seen that the geodesic open book on S^*S^2 is constructed in such a way that it is adapted to the geodesic flow of the round metric. On the other hand, by considering the action on S^3 of the subgroup $\{a_\theta\} \subset S^3$ of the previous section, we obtain an S^1-family $\{p_\theta : S^3\backslash a_\theta(L) \to S^1\}$ of open book decompositions on S^3 (here, L is the Hopf link). These are respectively adapted to the Reeb dynamics of α_θ, and start from the stabilized open book p_0 on S^3 (adapted to α_i by Ex. 10 above); they are all just rotations of each other.

Note that Prop. 2.18, the push-forward of p_0 under the Hopf map, i.e. $\overline{p}_0 := \pi_*(p_0) = p_0 \circ \pi^{-1}$: $S^*S^2\backslash B_0 \to S^1$ where B_0 is the disjoint union of the unit cotangent fibers over the north and south poles N, S in S^2 (i.e. the image of the Hopf link under π), is adapted to the infinite magnetic flow. The pages are cylinders obtained as follows: $S^*S^2\backslash B_0 \cong ((-1,1)\times S^1)\times S^1$ is a trivial bundle over $S^2\backslash\{N,S\} \cong (-1,1)\times S^1$ (the Euler class of S^*S^2 is -2), and $\overline{p}_0$ is the trivial fibration.

The push-forward $\overline{p}_\theta = \pi_*(p_\theta) : S^*S^2\backslash B_\theta \to S^1$ is then an open book decomposition on S^*S^2, which coincides with the geodesic open book at $\theta = \pi$. The binding B_θ consists of two magnetic geodesics for ω_θ; see Figure 5. We call any element of the family $\{\overline{p}_\theta\}$, a *magnetic open book decomposition*.

Digression: open books and Heegaard splittings. A 3-dimensional genus g (orientable) handlebody H_g is the 3-manifold with boundary resulting by taking the boundary connected sum of g-copies of the solid 2-torus $S^1 \times \mathbb{D}^2$ (here, we set $H_0 = B^3$ the 3-ball). H_g can also be obtained by

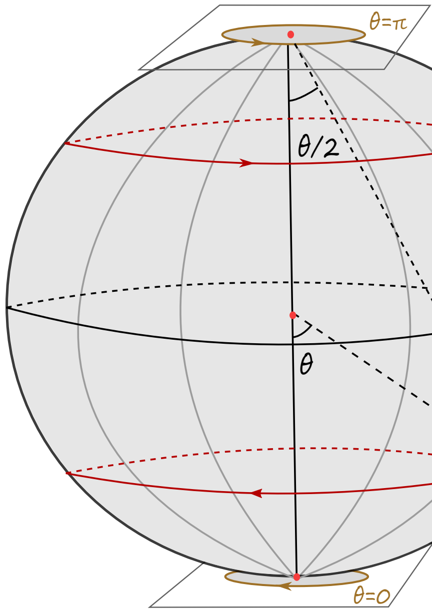

FIGURE 5. The binding of the magnetic open book $\overline{p}_\theta$ (in red), consisting of two circles of latitude θ and $\pi - \theta$, doubly covered by two Reeb orbits of α_θ. At $\theta = \pi$ the action of a_π maps the Hopf fiber over a point to the Hopf fiber over its antipodal (cf. [AGZ, Fig. 1]).

attaching a sequence of g 1-handles to B^3. Its boundary is Σ_g, the orientable surface of genus g. A *Heegaard splitting* of genus g of a closed 3-manifold X is a decomposition

$$X = H_g \bigcup_f H'_g,$$

where $f : \Sigma_g = \partial H_g \to \Sigma_g = \partial H'_g$ is a homeomorphism of the boundary of two copies of H_g. The surface Σ_g is called the splitting surface. Different choices of f in the mapping class group of Σ_g give, in general, different 3-manifolds. In fact, it is a fundamental theorem of 3-dimensional topology that every closed 3-manifold admits a Heegaard splitting. We have also touched upon another structural result for 3-manifolds: namely, that every closed 3-manifold admits an open book decomposition. Let us then discuss how to induce a Heegaard splitting from an open book.

Starting from a concrete open book decomposition $M \backslash B \to S^1 = \mathbb{R}/\mathbb{Z}$ of abstract type $M = \mathbf{OB}(P, \varphi)$, we obtain a Heegaard splitting via

$$H_g = \pi^{-1}([0, 1/2]) \cup B, \; H'_g = \pi^{-1}([1/2, 1]) \cup B,$$

where the splitting surface $\Sigma_g = P_0 \cup_B P_{1/2}$ is the double of the page $P_0 = \pi^{-1}(0)$, obtained by gluing P_0 to its "opposite" $P_{1/2} = \pi^{-1}(1/2)$. The gluing map f is simply given by φ on P_0, and the identity on $P_{1/2}$. Stabilizing the open book translates into a stabilization of the Heegaard splitting.

This shows that the Heegaard diagram thus induced is rather special, since the gluing map is trivial on "half" of the splitting surface. In fact, not every Heegaard splitting arises this way, as is

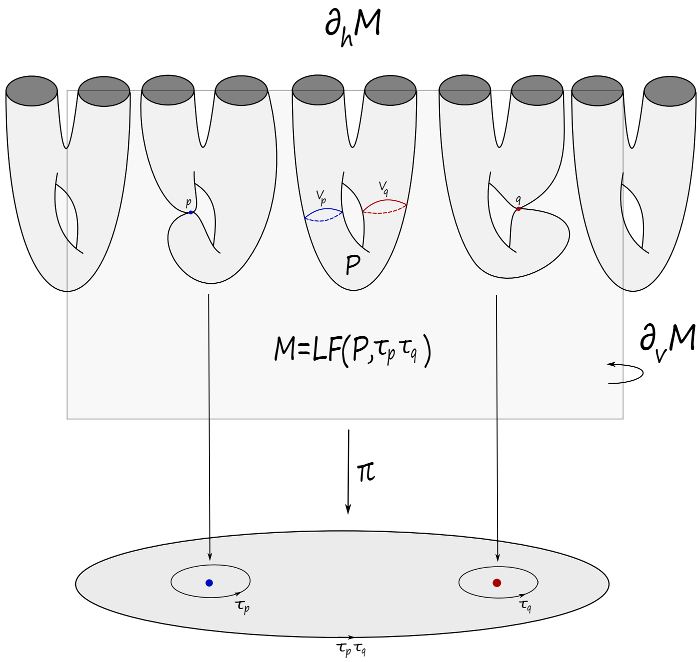

FIGURE 6. The Lefschetz fibration $\mathbf{LF}(P, \tau_p \tau_q)$ over $\mathbb{D}^2$.

easy to see (e.g. the lens spaces are precisely the 3-manifolds with Heegaard splittings of genus 1, but only the lens spaces discussed in Example 2.12 arise from an open book with annulus page, since its relative mapping class group is generated by the Dehn twist).

Digression: open books and Lefschetz fibrations/pencils. We now explore some further interplay between symplectic and algebraic geometry.

Definition 2.20 (Lefschetz fibration)**.** Let M be a compact, connected, oriented, smooth 4-manifold with boundary. A *Lefschetz fibration* on M is a smooth map $\pi : M \to S$, where S is a compact, connected, oriented surface with boundary, such that each critical point p of π lies in the interior of M and has a local complex coordinate chart $(z_1, z_2) \in \mathbb{C}^2$ centered at p (and compatible with the orientation of M), together with a local complex coordinate z near $\pi(p)$, such that $\pi(z_1, z_2) = z_1^2 + z_2^2$ in this chart.

In other words, each critical point has a local (complex) Morse chart, and is therefore non-degenerate. We then have finitely many critical points due to compactness of M. One may also (up to perturbation of π) assume that there is a single critical point on each fiber of π. The regular fibers are connected oriented surfaces with boundary, whereas the singular fibers are immersed oriented surfaces with a transverse self-intersection (or node). This singularity is obtained from nearby fibers by pinching a closed curve (the *vanishing cycle*) to a point. See Figure 6.

The boundary of a Lefschetz fibration splits into two pieces:

$$\partial M = \partial_h M \cup \partial_v M,$$

where

$$\partial_h M = \bigcup_{b \in S} \partial \pi^{-1}(b),\ \partial_v M = \pi^{-1}(\partial B).$$

By construction, $\partial_h M$ is a circle fibration over S, and $\partial_v M$ is a surface fibration over ∂S. If we focus on the case $S = \mathbb{D}^2$, the two-disk, denoting the regular fiber P and $B = \partial P$, we necessarily have that $\partial_h M$ is trivial as a fibration, and $\partial_v M$ is the mapping torus P_ϕ of some monodromy $\phi : P \to P$. Therefore

$$\partial M = \partial_h M \cup \partial_v M = B \times \mathbb{D}^2 \bigcup P_\phi = \mathbf{OB}(P, \phi).$$

Now, the monodromy ϕ is not arbitrary, since orientations here play a crucial role. While every element in the symplectic mapping class group of a surface is a product of powers of Dehn twists along some simple closed loops, it turns out that ϕ is necessarily a product of *positive* powers of Dehn twists (once orientations are all fixed). In fact, $\phi = \prod_{p \in \text{crit}(\pi)} \tau_p$, where $\tau_p = \tau_{V_p}$ is the positive (or right-handed) Dehn twist along the corresponding vanishing cycle $V_p \cong S^1 \subset P$. This can be algebraically encoded via the monodromy representation

$$\rho : \pi_1(\mathbb{D}^2 \backslash \text{critv}(\pi)) \to \text{MCG}(P, \partial P),$$

where $\text{critv}(\pi) = \{x_1, \dots, x_n\}$, $x_i = \pi(p_i)$, is the finite set of critical values of π. We have

$$\pi_1(\mathbb{D}^2 \backslash \{x_1, \dots, x_n\}) = \langle g_\partial, g_1, \dots, g_n : g_\partial = \prod_{i=1}^n g_i \rangle,$$

where g_i is a small loop around x_i and $g_\partial = \partial \mathbb{D}^2$, and ρ is defined via $\rho(g_i) = \tau_{V_{p_i}}$.

Exercise 12. Show that the monodromy of the local model $\pi : \mathbb{C}^2 \to \mathbb{C}$, $\pi(z_1, z_2) = z_1^2 + z_2^2$, viewed as a Lefschetz fibration with fiber the cylinder T^*S^1, is precisely the positive Dehn twist along the zero section $S^1 \subset T^*S^1$. Deduce that the monodromy representation behaves as claimed, for a general Lefschetz fibration. See Figure 7.

Reciprocally, a 4-dimensional Lefschetz fibration on M over $\mathbb{D}^2$ is abstractly determined by the data of the regular fiber P (a surface with non-empty boundary) and a collection of simple closed loops $V_1, \dots, V_n \subset P$. This determines a monodromy $\phi = \prod_{i=1}^n \tau_{V_i}$, a product of positive Dehn twists along the vanishing cycles V_i. The recipe to construct M works as follows: decompose $P = \mathbb{D}^2 \bigcup H_1 \cup \dots \cup H_k$ into a handle decomposition with a single 0-handle $\mathbb{D}^2$ and a collection of 2-dimensional 1-handles $H_1, \dots, H_k \cong \mathbb{D}^1 \times \mathbb{D}^1$. One starts with the trivial Lefschetz fibration $M_0 = \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{D}^2$ with disk fiber; and then one attaches (thickened) 4-dimensional 1-handles $H_i \times \mathbb{D}^2$ to M_0 to obtain the trivial Lefschetz fibration $M_1 = P \times \mathbb{D}^2 \to \mathbb{D}^2$ with fiber P. In order to add the singularities, one attaches one 4-dimensional 2-handle $H = \mathbb{D}^2 \times \mathbb{D}^2$ along $V_i \subset P \times \{1\} \subset \partial M_1$, viewed as the attaching sphere $V_i = S^1 \times \{0\} \subset S^1 \times \mathbb{D}^2 \subset \partial H$. At each step of the 2-handle attachments, we obtain a fibration with monodromy representation ρ_i extending ρ_{i-1} and satisfying $\rho_i(g_i) = \tau_{V_i}$, starting from the trivial representation $\rho_0 = \mathbb{1} : \pi_1(\mathbb{D}^2) = \{1\} \to \text{MCG}(P, \partial P)$. We denote the resulting manifold as $M = \mathbf{LF}(P, \phi)$, for which we have a handle description with handles of index $0, 1, 2$.

Remark 2.21. The notation $\mathbf{LF}(P, \phi)$, although simple, is a bit misleading: we need to remember the factorization of ϕ, since different factorizations lead in general to different smooth 4-manifolds. One should perhaps use $\mathbf{LF}(P; V_1, \dots, V_n)$ instead, although we hope this will not lead to confusion.

Having said that, we summarize this discussion in the following:

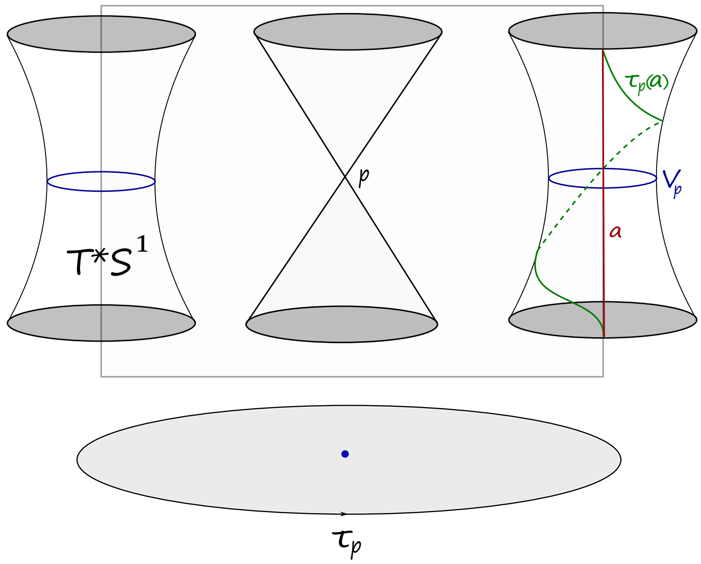

FIGURE 7. The local model for a Lefschetz singularity.

Lemma 2.22 (Relationship between Lefschetz fibrations and open books)**.** *We have*

$$\partial \mathbf{LF}(P, \phi) = \mathbf{OB}(P, \phi),$$

for $\phi = \prod_{i=1}^{n} \tau_{V_i}$ *a product of positive Dehn twists along a collection of vanishing cycles* $V_1, \ldots, V_n$ *in* P.

While so far this has been a discussion in the smooth category, one may upgrade this to the symplectic/contact category. While we have seen that open books support contact structures in the sense of Giroux, Lefschetz fibrations also support symplectic structures. This is encoded in the following:

Definition 2.23 (Symplectic Lefschetz fibrations)**.** An (exact) *symplectic* Lefschetz fibration on an exact symplectic 4-manifold $(M, \omega = d\lambda)$ is a Lefschetz fibration π for which the vertical and horizontal boundary are convex, and the fibers $\pi^{-1}(b)$ are symplectic with respect to ω, also with convex boundary.

Here, convexity means that the Liouville vector field is outwards pointing. Note that, by Stokes's theorem and exactness of ω, a symplectic Lefschetz fibration cannot have contractible vanishing cycles, since otherwise there would be a non-constant symplectic sphere in a fiber. The description of Lefschetz fibrations in terms of handle attachments can also be upgraded to the sympectic category via the notion of a *Weinstein handle*. After smoothing out the corner $\partial_h M \cap \partial_v M$, the boundary ∂M becomes contact-type via $\alpha = \lambda|_{\partial M}$, and the contact structure $\xi = \ker \alpha$ is supported by the open book at the boundary. The contact manifold $(\partial M, \xi)$ is said to be *symplectically filled* by (M, ω) (see the discussion below on symplectic fillings of contact manifolds).

Since the space of symplectic forms on a two-manifold is convex and hence contractible, one can show that, given the Lefschetz fibration $\mathbf{LF}(P, \phi)$, an *adapted* symplectic form (i.e. as in the definition above) exists and is unique up to symplectic deformation. Therefore, similarly as in

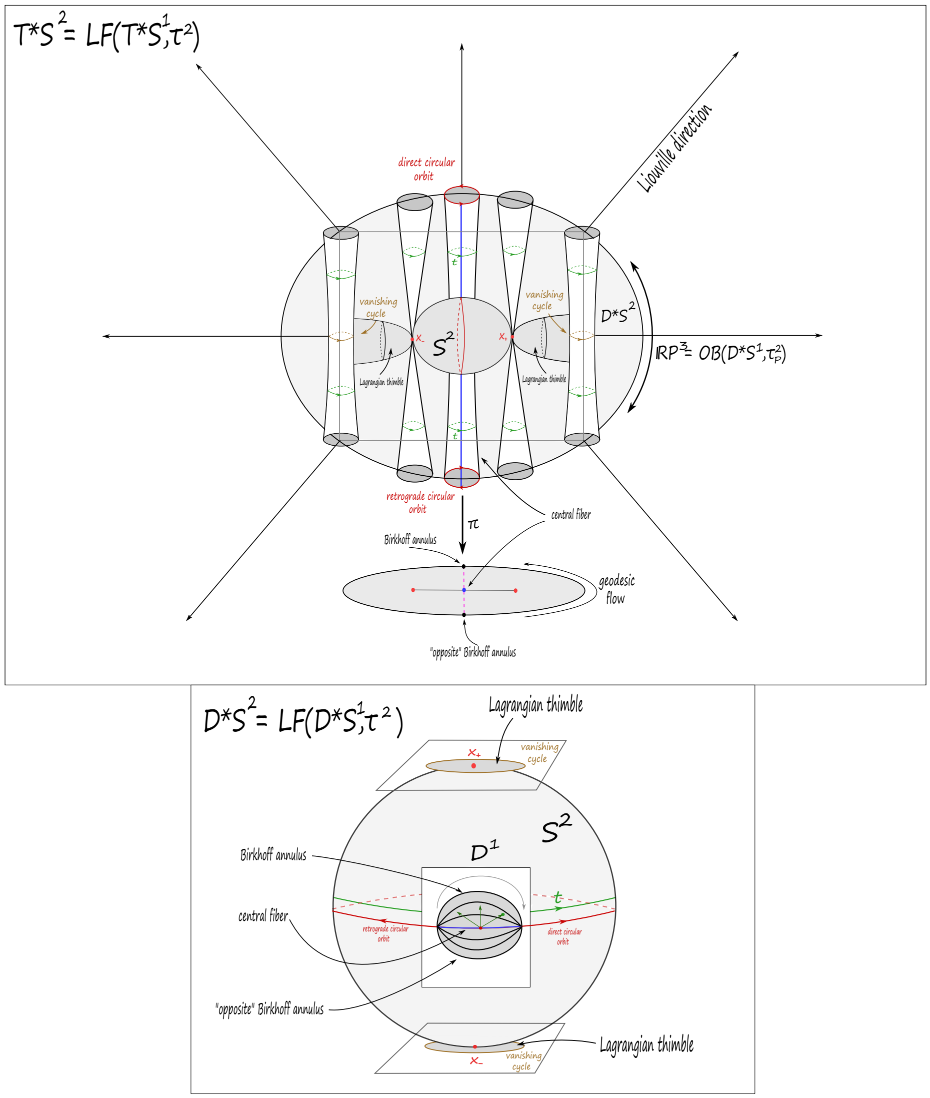

FIGURE 8. The standard Lefschetz fibration on $\mathbb{D}^*S^2 = \mathbf{LF}(\mathbb{D}^*S^1, \tau^2)$, where τ is the Dehn twist along the zero section $S^1 \subset \mathbb{D}^*S^1$. In the picture above, we draw T^*S^2, and the fibers on $\mathbb{D}^*S^2$ are obtained by projecting along the Liouville direction. These are drawn in the picture below. The two critical points induce the monodromy τ^2.

Giroux's correspondence, one can talk about $\mathbf{LF}(P, \phi)$ as a symplectomorphism class of symplectic manifolds.

Example 2.24. An example which is relevant for the spatial CR3BP is that of T^*S^2. We consider the *Brieskorn variety*

$$V_\epsilon = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : \sum_{j=0}^{n} z_j^2 = \epsilon\},$$

and the associated *Brieskorn manifold* $\Sigma_\epsilon = V_\epsilon \cap S^{2n+1}$. If $\epsilon = 0$, V_0 has an isolated singularity at the origin, and Σ_0 is called the *link of the singularity*. For $\epsilon \neq 0$, the domain $V_\epsilon^{cpt} = V_\epsilon \cap B^{2n+2}$ is a smooth manifold, with boundary $\Sigma_\epsilon \cong \Sigma_0$; the manifold V_ϵ also inherits a symplectic form by restriction of ω_{std} on $\mathbb{C}^{n+1}$. Similarly, Σ_ϵ inherits a contact form by restriction of the standard contact form $\alpha_{std} = i\sum_j z_j d\overline{z}_j - \overline{z}_j dz_j$. In fact, V_ϵ is a Stein manifold, and V_ϵ^{cpt} is a Stein filling of Σ_ϵ; see the discussion on Stein manifolds above, and fillings below.

Exercise 13. Prove that the map

$$(V_1, \omega_{std}) \to (T^*S^n \subset T^*\mathbb{R}^{n+1}, \omega_{can}),\ z = q + ip \mapsto (\|q\|^{-1}q, \|q\|p)$$

is a symplectomorphism, which restricts to a contactomorphism

$$(\Sigma_0, \alpha_{std}) \to (S^*S^n \subset T^*\mathbb{R}^{n+1}, \lambda_{can}).$$

The standard Lefschetz fibration on T^*S^n can be obtained from the Brieskorn variety model as

$$V_1 \to \mathbb{C},\ (z_0, \dots, z_n) \mapsto z_0.$$

This induces the geodesic open book on S^*S^n at the boundary, given by the same formula.

Exercise 14. Prove, at least for $n = 2$, that the above map induces the Lefschetz fibration $T^*S^2 = \mathbf{LF}(T^*S^1, \tau^2)$, where τ is the Dehn twist along the vanishing cycle $S^1 \subset T^*S^1$, the zero section. Conclude that $S^*S^2 = \mathbb{R}P^3 = \mathbf{OB}(\mathbb{D}^*S^1, \tau^2)$. See Figure 8.

To tie the above discussion with classical algebraic geometry, we introduce the following notion (in the closed case):

Definition 2.25 (Lefschetz pencil)**.** Let M be a closed, connected, oriented, smooth 4-manifold. A *Lefschetz pencil* on M is a Lefschetz fibration $\pi : M\backslash L \to \mathbb{C}P^1$,where $L \subset M$ is a finite collection of points, such that near each base point $p \in L$ there exists a complex coordinate chart (z_1, z_2) in which π looks like the Hopf map $\pi(z_1, z_2) = [z_1 : z_2]$.

Lefschetz pencils arise naturally in the study of projective varieties, and linear systems of line bundles over them. The basic construction is the following: Consider two distinct homogeneous polynomials $F(x, y, z), G(x, y, z)$ of degree d in projective coordinates $[x : y : z] \in \mathbb{C}P^2$ (i.e. sections of the holomorphic line bundle $\mathcal{O}(d)$), generic in the sense that $V(F) = \{F = 0\}$ and $V(G) = \{G = 0\}$ are smooth degree d curves, of genus $g = \frac{(d-1)(d-2)}{2}$ by the genus-degree formula, and so that the base locus $V(F) \cap V(G) = L$ consists of a collection of d^2 distinct points (by Bézout's theorem). Consider the *degree d pencil* $\{C_{[\lambda:\mu]}\}_{[\lambda:\mu]\in\mathbb{C}P^1}$, where

$$C_{[\lambda:\mu]} = V(\lambda F + \mu G) \subset \mathbb{C}P^2.$$

Through any point in $\mathbb{C}P^2\backslash L$, there is a unique $C_{[\lambda:\mu]}$ which contains it. We then have a Lefschetz pencil

$$\pi : \mathbb{C}P^2\backslash L \to \mathbb{C}P^1,$$

where $\pi([x : y : z]) = [\lambda : \mu]$ if $C_{[\lambda:\mu]}$ is the unique degree d curve in the family passing through $[x : y : z]$.

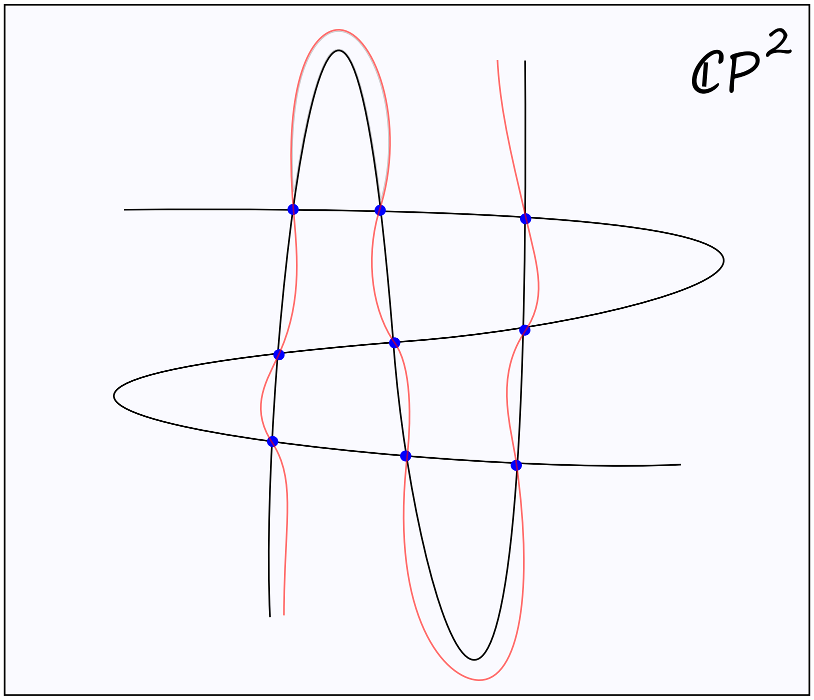

FIGURE 9. A cartoon of a pencil of cubics, where L consists of 9 points, and each fiber has genus 1.

By construction, every curve in the pencil meets at the d^2 points in L. One can further perform a complex blow-up along each of these points, by adding an exceptional divisor (a copy of $\mathbb{C}P^1$) of all possible incoming directions at a given point, and the result is a Lefschetz fibration

$$Bl_L\pi : Bl_L\mathbb{C}P^2 \to \mathbb{C}P^1.$$

By construction, this Lefschetz fibration has plenty of spheres, i.e. the exceptional divisors, which are sections of the fibration.

The above construction also extends to the case of closed 4-dimensional projective varieties in some ambient projective space. Moreover, as we have already mentioned, projective varieties are Kähler, and in particular symplectic. It is a very deep fact that the above construction extends beyond the algebraic case to the general case of *all* closed symplectic 4-manifolds:

Theorem I (Donaldson)**.** *Any closed symplectic 4-manifold (M,ω) admits Lefschetz pencils with symplectic fibers. In fact, if $[\omega] \in H^2(M;\mathbb{Z})$ is integral, the fibers are Poincaré dual to $k[\omega]$ for some sufficiently large $k \gg 0$.*

The above implies that techniques from algebraic geometry can also be applied in the symplectic category, and the interplay is very rich. From the above discussion, after blowing up a finite number of points on the given closed symplectic 4-manifold (M,ω), we obtain a Lefschetz fibration.

Digression: symplectic cobordisms and fillings. We have already seen the fundamental relationship between contact and symplectic geometry. We now touch upon this a bit further.

Definition 2.26 (Symplectic cobordism)**.** A (strong) symplectic cobordism from a closed contact manifold (X_-,ξ_-) to a closed contact manifold (X_+,ξ_-) is a compact symplectic manifold (M,ω) satisfying:

- $\partial M = X_+ \bigsqcup X_-$;
- $\omega = d\lambda_\pm$ is exact near $X_\pm$, and the (local) Liouville vector field $V_\pm$ (defined via $i_{V_\pm}\omega = \lambda_\pm$) is inwards pointing along X_- and outwards pointing along X_+;
- $\ker \lambda_\pm|_{X_\pm} = \xi_\pm$.

If $\omega = d\lambda$ is globally exact and the Liouville vector field is outwards/inwards pointing along $X_\pm$, we say that (M,ω) is a *Liouville* cobordism. The boundary component X_+ is called *convex* or *positive*, and X_-, *concave* or *negative*. Note that a symplectic cobordism is *directed*; in general there might be such a cobordism from X_- to X_+ but not viceversa. In fact, the relation $(X_-,\xi_-) \preceq (X_+,\xi_+)$ whenever there exists a symplectic cobordism as above, is reflexive, transitive, but *not* symmetric. We remark that the opposite convention on the choice of *to* and *from* are also used in the literature.

Exercise 15. Prove that $(X_-,\xi_-) \preceq (X_+,\xi_+)$ is reflexive and transitive.

Definition 2.27 (Symplectic filling/Liouville domain)**.** A (strong, Liouville) symplectic filling of a contact manifold (X,ξ) is a (strong, Liouville) compact symplectic cobordism from the empty set to (X,ξ). A Liouville filling is also called a *Liouville domain*.

The *Liouville manifold* associated to a Liouville domain (M,ω) is its *Liouville completion*, obtained by attaching a cylindrical end:

$$(\widehat{M},\widehat{\omega} = d\widehat{\lambda}) = (M,\omega = d\lambda) \cup_{\partial M} ([1,+\infty) \times \partial M, d(r\alpha)),$$

where $\alpha = \lambda|_{\partial M}$ is the contact formm at the boundary. Liouville manifolds are therefore "convex at infinity".

It is a fundamental question of contact topology whether a contact manifold is fillable or not, and, if so, how many fillings it admits (say, up to symplectomorphism, diffeomorphism, homeomorphism, homotopy equivalence, s-cobordism, h-cobordism,...). Note that, given a filling, one may choose to perform a symplectic blowup in the interior, which doesn't change the boundary but changes the symplectic manifold; in order to remove this trivial ambiguity one usually considers *symplectically aspherical* fillings, i.e. symplectic manifolds (M,ω) for which $[\omega]|_{\pi_2(M)} = 0$ (this holds if e.g. ω is exact, as the case of a Liouville filling).

For example, the standard sphere (S^{2n-1},ξ_{std}) admits the unit ball (B^{2n},ω_{std}) as a Liouville filling. A fundamental theorem of Gromov [Gro85, p. 311] says that this is unique (strong, symplectically aspherical:=ssa) filling up to symplectomorphism in dimension 4; this is known up to diffeomorphism in higher dimensions by a result of Eliashberg–Floer–McDuff [M91], but unknown up to symplectomorphism. This was generalized to the case of *subcritically* Stein fillable contact manifolds in [BGZ]. Another example is a unit cotangent bundle (S^*Q,ξ_{std}), which admits the standard Liouville filling $(\mathbb{D}^*Q,\omega_{std})$. There are known examples of manifolds Q with (S^*Q,ξ_{std}) admitting only one ssa filling up to symplectomorphism (e.g. $Q=\mathbb{T}^2$, [Wen]; if $n\geq 3$ and $Q=\mathbb{T}^n$, this also holds up to diffeomorphism [BGM, GKZ]), but there are other examples with non-unique ssa fillings which are not blowups of each other (e.g. $Q=S^n$, $n\geq 3$ [Oba]). See also [SvHM, LMY, LO]. The literature on fillings is vast (especially in dimension 3) and this list is by all means non-exhaustive.

Remark 2.28. There are also other notions of symplectic fillability: weak, Stein, Weinstein... which we will not touch upon. The set of contact manifolds admitting a filling of every such type is related via the following inclusions:

$$\{\text{Stein}\} \subset \{\text{Weinstein}\} \subset \{\text{Liouville}\} \subset \{\text{strong}\} \subset \{\text{weak}\}.$$

The first inclusion is an equality by a deep result of Eliashberg [CieEli]. All others are strict inclusions, something that has been in known in dimension 3 for some time [Bow, Ghi, Eli96], but has been fully settled in higher-dimensions only very recently [BGM, BCS, ZZ, MNW].

A very broad class for which very strong uniqueness results hold is the following. We say that a contact 3-manifold (X, ξ) is *planar* if ξ is supported (in the sense of Giroux) by an open book whose page has genus zero.

Theorem J (Wendl [Wen])**.** *Assume that (M, ω) is a strong symplectic filling of a planar contact 3-manifold (X, ξ), and fix a supporting open book of genus zero pages, i.e. $M = \mathbf{OB}(P, \phi)$ with $g(P) = 0$. Then (M, ω) is symplectomorphic to a (symplectic) blow-up of the symplectic Lefschetz fibration $\mathbf{LF}(P, \phi)$.*

If we assume that the strong filling is *minimal*, in the sense that it doesn't have symplectic spheres of self-intersection -1 (i.e. exceptional divisors), such filling is then uniquely determined. It follows as a corollary, that a planar contact manifold is strongly fillable if and only if every supporting planar open book has monodromy isotopic to a product of positive Dehn twists. This reduces the study of strong fillings of a planar contact 3-manifolds to the study of factorizations of a given monodromy into product of positive Dehn twists, a problem of geometric group theory in the mapping class group of a genus zero surface.

References. A good introductory textbook to contact topology is Geiges' book [G08]. For an introduction to symplectic topology, McDuff-Salamon [MS] is a must-read. Anna Cannas da Silva [CdS] is also a very good source, touching on Kähler geometry as well as toric geometry, relevant for the classical theory of integrable systems. For open books and Giroux's correspondence in dimension 3, Etnyre's notes [E06] is a good place to learn. For open books in complex singularity theory (i.e. Milnor fibrations), the classical book by Milnor [M68] is a gem. For related reading on Brieskorn manifolds in contact topology, Lefschetz fibrations and further material, Kwon-van Koert [KvK] is a great survey. Another good source for symplectic geometry in dimension 4, Lefschetz pencils, and its relationship to holomorphic curves and rational/ruled surfaces, is Wendl's recent book [Wen2].

3. The three-body problem.

After paving the way, we now discuss a very old conundrum. The setup of the classical 3-body problem consists of three bodies in $\mathbb{R}^3$, subject to the gravitational interactions between them, which are governed by Newton's laws of motion. Given initial positions and velocities, the problem consists in predicting the future positions and velocities of the bodies. The understanding of the resulting dynamical system is quite a challenge, and an outstanding open problem.

We consider three bodies: Earth (E), Moon (M) and Satellite (S), with masses m_E, m_M, m_S. We have the following special cases:

- (restricted) $m_S = 0$ (the Satellite is negligible wrt the *primaries* E and M);
- (circular) Each primary moves in a circle, centered around the common center of mass of the two (as opposed to general ellipses);
- (planar) S moves in the plane containing the primaries;
- (spatial) The planar assumption is dropped, and S is allowed to move in three-space.

The restricted problem then consists in understanding the dynamics of the trajectories of the Satellite, whose motion is affected by the primaries, but not vice-versa. For simplicity, we will use the acronym CR3BP=circular restricted three-body problem. We denote the *mass ratio* by $\mu = \frac{m_M}{m_E+m_M} \in [0,1]$, and we normalize so that $m_E + m_M = 1$, and so $\mu = m_M$.

In a suitable inertial plane spanned by the E and M, the position of the Earth becomes $E(t) = (\mu\cos(t), \mu\sin(t))$, and the position of the Moon is $M(t) = (-(1-\mu)\cos(t), -(1+\mu)\sin(t))$. The time-dependent Hamiltonian whose Hamiltonian dynamics we wish to study is then

$$H_t : \mathbb{R}^3 \backslash \{E(t), M(t)\} \to \mathbb{R}$$

$$H_t(q,p) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q - M(t)\|} - \frac{1-\mu}{\|q - E(t)\|},$$

i.e. the sum of the kinetic energy plus the two Couloumb potentials associated to each primary. Note that this Hamiltonian is time-dependent. To remedy this, we choose rotating coordinates, in which both primaries are at rest; the price to pay is the appearance of angular momentum term in the Hamiltonian which represents the centrifugal and Coriolis forces in the rotating frame. Namely, we undo the rotation of the frame, and assume that the positions of Earth and Moon are $E = (\mu, 0, 0)$, $M = (-1+\mu, 0, 0)$. After this (time-dependent) change of coordinates, which is just the Hamiltonian flow of $L = p_1q_2 - p_2q_1$, the Hamiltonian becomes

$$H : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}$$

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q - M\|} - \frac{1-\mu}{\|q - E\|} + p_1q_2 - p_2q_1,$$

and in particular is *autonomous*. By preservation of energy, this means that it is a preserved quantity of the Hamiltonian motion. The planar problem is the subset $\{p_3 = q_3 = 0\}$, which is clearly invariant under the Hamiltonian dynamics.

There are precisely five critical points of H, called the *Lagrangian points* L_i, $i = 1, \dots, 5$, ordered so that $H(L_1) < H(L_2) < H(L_3) < H(L_4) = H(L_5)$ (in the case $\mu < 1/2$; if $\mu = 1/2$ we further have $H(L_2) = H(L_3)$). L_1, L_2, L_3, all saddle points, lie in the axis between Earth and Moon (they are the *collinear* Lagrangian points). L_1 lies between the latter, while L_2 on the opposite side of the Moon, and L_3 on the opposite side of the Earth. The others, L_4, L_5, are maxima, and are called the *triangular* Lagrangian points. For $c \in \mathbb{R}$, consider the energy hypersurface $\Sigma_c = H^{-1}(c)$. If

$$\pi : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}^3 \backslash \{E, M\},\ \pi(q,p) = q,$$

is the projection onto the position coordinate, we define the *Hill's region* of energy c as

$$\mathcal{K}_c = \pi(\Sigma_c) \in \mathbb{R}^3 \backslash \{E, M\}.$$

This is the region in space where the Satellite of energy c is allowed to move. If $c < H(L_1)$ lies below the first critical energy value, then $\mathcal{K}_c$ has three connected components: a bounded one around the Earth, another bounded one around the Moon, and an unbounded one. Namely, if the Satellite starts near one of the primaries, and has low energy, then it stays near the primary also in the future. The unbounded region corresponds to asteroids which stay away from the primaries. Denote the first two components by $\mathcal{K}_c^E$ and $\mathcal{K}_c^M$, as well as $\Sigma_c^E = \pi^{-1}(\mathcal{K}_c^E) \cap \Sigma_c$, $\Sigma_c^M = \pi^{-1}(\mathcal{K}_c^M) \cap \Sigma_c$, the components of the corresponding energy hypersurface over the bounded components of the Hill region. As c crosses the first critical energy value, the two connected components $\mathcal{K}_c^E$ and $\mathcal{K}_c^M$ get glued to each other into a new connected component $\mathcal{K}_c^{E,M}$, which topologically is their connected sum. Then, the Satellite in principle has enough energy to transfer between Earth and Moon. In

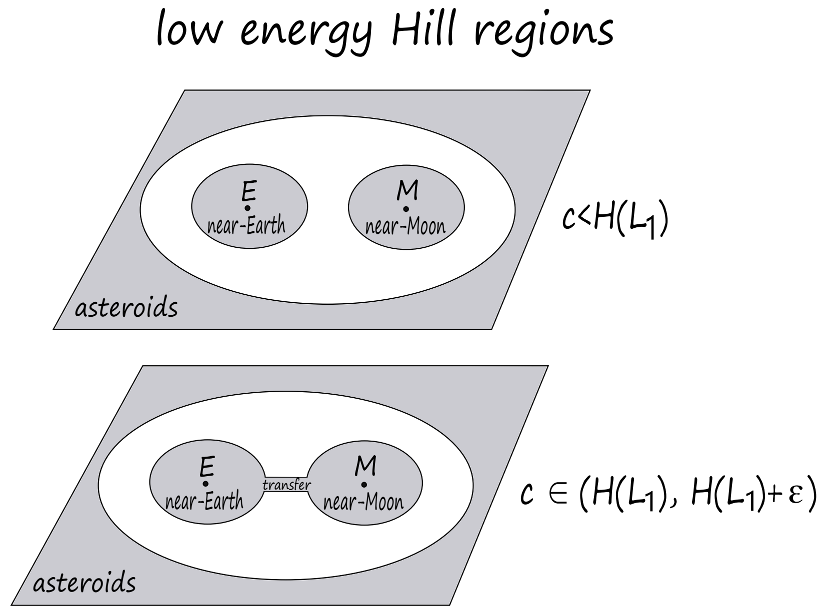

FIGURE 10. The low energy Hill regions.

terms of Morse theory, crossing critical values corresponds precisely to attaching handles, so similar handle attachments occur as we sweep through the energy values until the Hill region becomes all of position space. See Figure 10.

4. MOSER REGULARIZATION.

The 5-dimensional energy hypersurfaces are non-compact, due to collisions of the massless body S with one of the primaries, i.e. when if $q = M$ or $q = E$. Note that the Hamiltonian becomes singular at collisions because of the Couloumb potentials, and conservation of energy implies that the momenta necessarily explodes whenever S collides (i.e. $p = \infty$). Fortunately, there are ways to regularize the dynamics even after collision. Intuitively, the effect is: whenever S collides with a primary, it bounces back to where it came from, and hence we continue the dynamics beyond the catastrophe. More formally, one is looking for a compactification of the energy hypersurface, which may be viewed as the level set of a new Hamiltonian on another symplectic manifold, in such a way that the Hamiltonian dynamics of the compact, regularized level set is a *reparametrization* of the original one (time is forgotten under regularization).

Two body collisions can be regularized via Moser's recipe. This consists in interchanging position and momenta, and compactifying by adding a point at infinity corresponding to collisions (where the velocity explodes). The bounded components Σ_c^E and Σ_c^M (for $c < H(L_1)$), as well as $\Sigma_c^{E,M}$ (for $c \in (H(L_1), H(L_1) + \epsilon)$), are thus compactified to compact manifolds $\overline{\Sigma}_c^E$, $\overline{\Sigma}_c^M$, and $\overline{\Sigma}_c^{E,M}$. The first two are diffeomorphic to $S^*S^3 = S^3 \times S^2$, and should be thought of as level sets in (two different copies of) (T^*S^3, ω_{std}) of a suitable regularized Hamiltonian $Q : T^*S^3 \to \mathbb{R}$. The fiber of the level sets $\overline{\Sigma}_c^E$, $\overline{\Sigma}_c^M$ over (a momenta) $p \in S^3$ is a 2-sphere allowed positions q in order to have fixed energy. If $p = \infty$ is the North pole, the fiber, called the *collision locus*, is the result of a real blow-up at a primary, i.e. we add all possible "infinitesimal" positions nearby (which one may think of as all unit directions in the tangent space of the primary). On the other hand, $\overline{\Sigma}_c^E$ is a copy of $S^*S^3\#S^*S^3$, which can be understood in terms of handle attachments along a critical point of index 1. In the planar problem, the situation is similar: we obtain copies of $S^*S^2 = \mathbb{R}P^3$ and $\mathbb{R}P^3\#\mathbb{R}P^3$.

In terms of formulas, this can be done as follows.

4.1. **Stark-Zeeman systems.** We will only do the subcritical case $c < H(L_1)$. By restricting the Hamiltonian to the Earth or Moon component, we can view the three-body problem as a *Stark-Zeeman* system, which is a more general class of mechanical systems.

To define such systems in general, consider a twisted symplectic form

$$\omega = d\vec{p} \wedge d\vec{q} + \pi^*\sigma_B,$$

with $\sigma_B = \frac{1}{2}\sum B_{ij} dq_i \wedge dq_j$ a 2-form on the position variables (a *magnetic* term, which physically represents the presence of an electromagnetic field, as in Maxwell's equations), and $\pi(q,p) = q$ the projection to the base. A *Stark-Zeeman system* for such a symplectic form is a Hamiltonian of the form

$$H(\vec{q},\vec{p}) = \frac{1}{2}\|\vec{p}\|^2 + V_0(\vec{q}) + V_1(\vec{q}),$$

where $V_0(\vec{q}) = -\frac{g}{\|\vec{q}\|}$ for some positive coupling constant g, and V_1 is an extra potential.[1]

We will make two further assumptions.

Assumptions.

(A1) We assume that the magnetic field is exact with primitive 1-form $\vec{A}$. Then with respect to $d\vec{p} \wedge d\vec{q}$ we can write

$$H(\vec{q},\vec{p}) = \frac{1}{2}\|\vec{p} + \vec{A}(\vec{q})\|^2 + V_0(\vec{q}) + V_1(\vec{q}).$$

(A2) We assume that $\vec{A}(\vec{q}) = (A_1(q_1,q_2), A_2(q_1,q_2), 0)$, and that the potential satisfies that symmetry $V_1(q_1,q_2,-q_3) = V_1(q_1,q_2,q_3)$.

Observe that these assumptions imply that the planar problem, defined as the subset $\{(\vec{q},\vec{p}) : q_3 = p_3 = 0\}$, is an invariant set of the Hamiltonian flow. Indeed, we have

$$\dot{q}_3 = \frac{\partial H}{\partial p_3} = p_3, \text{ and } \dot{p}_3 = -\frac{\partial H}{\partial q_3} = -\frac{g q_3}{\|\vec{q}\|^3} - \frac{\partial V_1}{\partial q_3}. \tag{4.3}$$

Both these terms vanish on the subset $q_3 = p_3 = 0$ by noting that the symmetry implies that $\frac{\partial V_1}{\partial q_3}|_{q_3=0} = 0$.

For non-vanishing g, Stark-Zeeman systems have a singularity corresponding to two-body collisions, which we will regularize by Moser regularization. To do so, we will define a new Hamiltonian Q on T^*S^3 whose dynamics correspond to a reparametrization of the dynamics of H. We will describe the scheme for energy levels $H = c$, which we need to *fix* a priori (i.e. the regularization is not in principle for all level sets at once). Define the intermediate Hamiltonian

$$K(\vec{q},\vec{p}) := (H(\vec{q},\vec{p}) - c)\|\vec{q}\|.$$

[1]In this section, we will use the symbol $\vec{\ }$ for vectors in $\mathbb{R}^3$ to make our formulas for Moser regularization simpler. We will use the convention that $\xi \in \mathbb{R}^4$ has the form $(\xi_0, \vec{\xi})$.

For $\vec{q} \neq 0$, this function is smooth, and its Hamiltonian vector field equals

$$X_K = \|\vec{q}\| \cdot X_H + (H - c)X_{\|\vec{q}\|}.$$

We observe that X_K is a multiple of X_H on the level set $K = 0$. Writing out K gives

$$K = \left(\frac{1}{2}(\|\vec{p}\|^2 + 1) - (c + 1/2) + \langle \vec{p}, \vec{A} \rangle + \frac{1}{2}\|\vec{A}\|^2 + V_1(\vec{q})\right) \|\vec{q}\| - g.$$

Stereographic projection. We now substitute with the stereographic coordinates. The basic idea is to switch the role of momentum and position in the $\vec{q}, \vec{p}$-coordinates, and use the $\vec{p}$-coordinates as position coordinates in $T^*\mathbb{R}^n$ (for any n), where we think of $\mathbb{R}^n$ as a chart for S^n. We set

$$\vec{x} = -\vec{p}, \quad \vec{y} = \vec{q}.$$

We view T^*S^n as a symplectic submanifold of $T^*\mathbb{R}^{n+1}$, via

$$T^*S^n = \{(\xi, \eta) \in T^*\mathbb{R}^{n+1} | \ \|\xi\|^2 = 1, \ \langle \xi, \eta \rangle = 0\}.$$

Let $N = (1, 0, \dots, 0) \in S^n$ be the north pole. To go from $T^*S^n \backslash T^*_N S^n$ to $T^*\mathbb{R}^n$ we use the stereographic projection, given by

$$\begin{aligned} \vec{x} &= \frac{\vec{\xi}}{1 - \xi_0} \\ \vec{y} &= \eta_0 \vec{\xi} + (1 - \xi_0)\vec{\eta}. \end{aligned} \tag{4.4}$$

To go from $T^*\mathbb{R}^n$ to $T^*S^n \backslash T^*_N S^n$, we use the inverse given by

$$\begin{aligned} \xi_0 &= \frac{\|\vec{x}\|^2 - 1}{\|\vec{x}\|^2 + 1} \\ \vec{\xi} &= \frac{2\vec{x}}{\|\vec{x}\|^2 + 1} \\ \eta_0 &= \langle \vec{x}, \vec{y} \rangle \\ \vec{\eta} &= \frac{\|\vec{x}\|^2 + 1}{2}\vec{y} - \langle \vec{x}, \vec{y} \rangle \vec{x}. \end{aligned} \tag{4.5}$$

These formulas imply the following identities

$$\frac{2}{\|\vec{x}\|^2 + 1} = 1 - \xi_0, \quad \|\vec{y}\| = \frac{2\|\eta\|}{\|\vec{x}\|^2 + 1} = (1 - \xi_0)\|\eta\|$$

which allows us to simplify the expression for K. Setting $n = 3$, we obtain a Hamiltonian $\tilde{K}$ defined on T^*S^3, given by

$$\begin{aligned} \tilde{K} &= \left(\frac{1}{1 - \xi_0} - (c + 1/2) - \frac{1}{1 - \xi_0}\langle \vec{\xi}, \vec{A}(\xi, \eta) \rangle + \frac{1}{2}\|\vec{A}(\xi, \eta)\|^2 + V_1(\xi, \eta)\right)(1 - \xi_0)\|\eta\| - g \\ &= \|\eta\| \left(1 - (1 - \xi_0)(c + 1/2) - \langle \vec{\xi}, \vec{A}(\xi, \eta) \rangle + (1 - \xi_0)\left(\frac{1}{2}\|\vec{A}(\xi, \eta)\|^2 + V_1(\xi, \eta)\right)\right) - g \end{aligned}$$

Put

$$\begin{aligned} f(\xi, \eta) &= 1 + (1 - \xi_0)\left(-(c + 1/2) + \frac{1}{2}\|\vec{A}(\xi, \eta)\|^2 + V_1(\xi, \eta)\right) - \langle \vec{\xi}, \vec{A}(\xi, \eta) \rangle \\ &= 1 + (1 - \xi_0)b(\xi, \eta) + M(\xi, \eta) \end{aligned} \tag{4.6}$$

where

$$b(\xi,\eta) = -(c+1/2) + \frac{1}{2}\|\vec{A}(\xi,\eta)\|^2 + V_1(\xi,\eta)$$
$$M(\xi,\eta) = -\langle \vec{\xi}, \vec{A}(\xi,\eta)\rangle$$

Note that the collision locus corresponds to $\xi_0 = 1$, i.e. the cotangent fiber over N. The notation is supposed to suggest that $(1-\xi_0)b(\xi,\eta)$ vanishes on the collision locus and M is associated with the magnetic term; it is not the full magnetic term, though. We then have that

$$\tilde{K} = \|\eta\| f(\xi,\eta) - g.$$

To obtain a smooth Hamiltonian, we define the Hamiltonian

$$Q(\xi,\eta) := \frac{1}{2} f(\xi,\eta)^2 \|\eta\|^2.$$

The dynamics on the level set $Q = \frac{1}{2}g^2$ are a reparametrization of the dynamics of $\tilde{K} = 0$, which in turn correspond to the dynamics of $H = c$.

Remark 4.1. We have chosen this form to stress that Q is a deformation of the Hamiltonian describing the geodesic flow on the round sphere, which is given by level sets of the Hamiltonian

$$Q_{round} = \frac{1}{2}\|\eta\|^2.$$

This is the dynamics that one obtains in the regularized Kepler problem (the two-body problem; see below), corresponding to the Reeb dynamics of the contact form given by the standard Liouville form. As we have seen, this is a Giroux form for the open book $S^*S^3 = \mathbf{OB}(\mathbb{D}^*S^2, \tau^2)$, supporting the standard contact structure on S^*S^3.

Formula for the restricted three-body problem. Since the restricted three-body problem is our main interest, we conclude this section by giving the explicit formula for this problem. By completing the squares, we obtain

$$H(\vec{q},\vec{p}) = \frac{1}{2}\left((p_1+q_2)^2 + (p_2-q_1)^2 + p_3^2\right) - \frac{\mu}{\|\vec{q}-\vec{m}\|} - \frac{1-\mu}{\|\vec{q}-\vec{e}\|} - \frac{1}{2}(q_1^2+q_2^2).$$

This is then a Stark-Zeeman system with primitive

$$\vec{A} = (q_2, -q_1, 0),$$

coupling constant $g = \mu$, and potential

$$V_1(\vec{q}) = -\frac{1-\mu}{\|\vec{q}-\vec{e}\|} - \frac{1}{2}(q_1^2+q_2^2), \tag{4.7}$$

both of which satisfy Assumptions (A1) and (A2).

After a computation, we obtain

$$f(\xi,\eta) = 1 + (1-\xi_0)\left(-(c+1/2) + \xi_2\eta_1 - \xi_1\eta_2\right) - \xi_2(1-\mu) - \frac{(1-\mu)(1-\xi_0)}{\|\vec{\eta}(1-\xi_0) + \vec{\xi}\eta_0 + \vec{m} - \vec{e}\|}, \tag{4.8}$$

and we have

$$b(\xi,\eta) = -(c+1/2) - \frac{(1-\mu)}{\|\vec{\eta}(1-\xi_0) + \vec{\xi}\eta_0 + \vec{m} - \vec{e}\|} \tag{4.9}$$
$$M(\xi,\eta) = (1-\xi_0)(\xi_2\eta_1 - \xi_1\eta_2) - \xi_2(1-\mu). \tag{4.10}$$

4.2. **Levi-Civita regularization.** We follow the exposition in [FvK18]. Consider the map

$$\mathcal{L} : \mathbb{C}^2 \backslash (\mathbb{C} \times \{0\}) \to T^*\mathbb{C} \backslash \mathbb{C},$$
$$(u, v) \mapsto \left(\frac{u}{v}, 2v^2 \right),$$

where we view $\mathbb{C} \subset T^*\mathbb{C}$ as the zero section. Using $\mathbb{C}$ as a chart for S^2 via the stereographic projection along the north pole, this map extends to a map

$$\mathcal{L} : \mathbb{C}^2 \backslash \{0\} \to T^*S^2 \backslash S^2,$$

which is a degree 2 cover. Writing (p, q) for coordinates on $T^*\mathbb{C} = \mathbb{C} \times \mathbb{C}$ (this is the *opposite* to the standard convention, and comes from the Moser regularization), the Liouville form on $T^*\mathbb{C}$ is $\lambda = q_1 dp_1 + q_2 dp_2$, with associated Liouville vector field $X = q_1 \partial_{q_1} + q_2 \partial_{q_2}$. One checks that

$$\mathcal{L}^* \lambda = 2(v_1 du_1 - u_1 dv_1 + v_2 du_2 - u_2 dv_2),$$

whose derivative is the symplectic form

$$\omega = d\lambda = 4(dv_1 \wedge du_1 + dv_2 \wedge du_2).$$

Note that λ and ω are *different* from the standard Liouville and symplectic forms (resp.) on $\mathbb{C}^2$. However, the associated Liouville vector field defined via $i_V \omega = \lambda$ coincides with the standard Liouville vector field

$$V = \frac{1}{2}(u_1 \partial_{u_1} + u_2 \partial_{u_2} + v_1 \partial_{v_1} + v_2 \partial_{v_2}),$$

and we have $\mathcal{L}^* X = V$. We conclude:

Lemma 4.2. *A closed hypersurface $\Sigma \subset T^*S^2$ is fiber-wise star-shaped if and only if $\mathcal{L}^{-1}(\Sigma) \subset \mathbb{C}^2 \backslash \{0\}$ is star-shaped.*

Note that $\Sigma \cong S^*S^2 \cong \mathbb{R}P^3$, and $\mathcal{L}^{-1}(\Sigma) \cong S^3$, and so $\mathcal{L}$ induces a two-fold cover between these two hypersurfaces.

4.3. **Kepler problem.** We now work out the Moser and Levi-Civita regularizations of the *Kepler problem* at energy $-\frac{1}{2}$. This is the well-known two-body problem, whose Hamiltonian is given by

$$E : T^*(\mathbb{R}^2 \backslash \{0\}) \to \mathbb{R},$$
$$E(q, p) = \frac{1}{2}\|p\|^2 - \frac{1}{\|q\|}.$$

The result of Moser regularization is the Hamiltonian

$$K(p, q) = \frac{1}{2} \left(\|q\| \left(E(-q, p) + \frac{1}{2} \right) + 1 \right)^2 = \frac{1}{2} \left(\frac{1}{2} \left(\|p\|^2 + 1 \right) \|q\| \right)^2.$$

This is the kinetic energy of the "momentum" q, with respect to the round metric, viewed in the stereographic projection chart. It follows that its Hamiltonian flow is the round geodesic flow. Moreover, we have

$$X_K|_{E^{-1}(-1/2)}(p, q) = \|q\| X_E|_{E^{-1}(-1/2)}(-q, p),$$

so that the Kepler flow is a reparametrization of the round geodesic flow.

To understand the Levi-Civita regularization, we consider the shifted Hamiltonian $H = E + \frac{1}{2}$ (which has the same Hamiltonian dynamics). After substituing variables via the Levi-Civita map $\mathcal{L}$, we obtain

$$H(u,v) = \frac{\|u\|^2}{2\|v\|^2} - \frac{1}{2\|v\|^2} + \frac{1}{2}.$$

We then consider the Hamiltonian

$$Q(u,v) = \|v\|^2 H(u,v) = \frac{1}{2}(\|u\|^2 + \|v\|^2 - 1).$$

The level set $Q^{-1}(0) = H^{-1}(0)$ is the 3-sphere, and the Hamiltonian flow of Q, a reparametrization of that of H, is the flow of two uncoupled harmonic oscillators. This is precisely the Hopf flow. We summarize this discussion in the following:

Proposition 4.3. *The Moser regularization of the Kepler problem is the geodesic flow on S^2. Its Levi-Civita regularization is the Hopf flow on S^3, i.e. the double cover of the geodesic flow on S^2 (cf. Rk. 2.19).*

5. The perturbative philosophy, and some historical remarks.

One of the most basic approaches that underlies mathematics and physics is the perturbative approach. Basically, it means understanding a simplified situation first, where everything can be explicitly understood, and attempt to understand "nearby" situations by perturbing the parameters relevant to the problem in question.

In the context of celestial/classical mechanics, this was precisely the approach of Poincaré. The idea is to start with a limit case, which is *completely integrable* (i.e. an integrable system), perturb it, and study what remained. Integrable systems, roughly speaking, are those which allow enough symmetries so that the solutions to the equations of motion can be explicitly solved for. The solutions tend to admit descriptions in terms of algebraic geometry. In the classical setting of celestial mechanics, if phase-space is $2n$-dimensional and the Hamiltonian H Poisson-commutes with other $n-1$ Hamiltonians (which are therefore preserved under the Hamiltonian flow of H), the well-known Arnold-Liouville theorem provides action-angle coordinates in which the symplectic manifold is foliated by flow-invariant tori, along which the Hamiltonian flow is linear, with varying slopes (the *frequencies*). The generic tori are half-dimensional (and *Lagrangian*, i.e. the symplectic form vanishes along them), whereas there might also be degenerate lower-dimensional tori. This is the natural realm of toric symplectic geometry, dealing with symplectic manifolds which admit a Hamiltonian action of the torus, and the study of the corresponding moment maps and their associated Delzant polytopes. There is also a related theory in algebraic geometry, where the polytope is replaced with a fan.

The study of what remains after a small perturbation of an integrable system is the realm of KAM theory, as well as complementary weaker versions such as Aubry-Mather theory. Roughly speaking, the original version of the KAM theorem (due to Kolmogorov-Arnold-Moser) says that if one perturbs a "sufficiently irrational" Liouville torus, i.e. the vector of frequencies of the action is very badly approximated by rational numbers (it is *diophantine*) and moreover the Hessian with respect to angle variables is non-degenerate, then the Liouville tori survives to an invariant tori whose frequencies are close to the original one, and hence is foliated by orbits which are *quasi-periodic*, in the sense that they are dense in the tori and never close up. Aubry-Mather theory is meant to deal with the rest of the tori, including resonant ones which are foliated by closed orbits and non-diophantine non-resonant ones, as well as large deformations (as opposed to sufficiently small perturbations). This theory provides invariant subsets which are usually Cantor-like, and obtained via measure-theoretical means (they are the supports of invariant measures minimizing certain action functionals).

The Poincaré-Birkhoff theorem, and the planar three-body problem. The problem of finding closed orbits in the planar case of the restricted three-body problem goes back to ground-breaking work in celestial mechanics of Poincaré [P12, P87], building on work of G.W. Hill on the lunar problem [H77, H78]. The basic scheme for his approach may be reduced to:

(1) Finding a global surface of section for the dynamics;
(2) Proving a fixed point theorem for the resulting first return map.

This is the setting for the celebrated Poincaré-Birkhoff theorem, proposed and confirmed in special cases by Poincaré and later proved in full generality by Birkhoff in [Bi13]. The statement can be summarized as: if $f : A \to A$ is an area-preserving homeomorphism of the annulus $A = [-1, 1] \times S^1$ that satisfies a *twist* condition at the boundary (i.e. it rotates the two boundary components in opposite directions), then it admits infinitely many periodic points of arbitrary large period. The fact that the area is preserved is a consequence of Liouville's theorem for Hamiltonian systems; we have basically used this in our proof of Proposition 2.16.

The whole point of a global surface of section is to reduce a *continuous* flow on a 3-manifold to the *discrete* dynamics of a map on a 2-manifold, thus reducing by one the degrees of freedom. It is perhaps fair to say, that this key (and beautiful) idea is responsible for motivating the well-studied area of dynamics on surfaces, a huge industry in its own right.

The direct and retrograde orbits. The actual physical Moon is in *direct* motion around the Earth (i.e. it rotates in the same direction around the Earth as the Earth around the Sun). The opposite situation is a *retrograde* motion. In [H77, H78], while attempting to model the motion of the Moon, Hill indeed finds both direct and retrograde orbits. While still an idealized situation, such direct orbit is a reasonable approximation to the actual orbit of the Moon, and Hill even goes further to find better approximations via perturbation theory, something which deeply impressed Poincaré himself. Topologically, one may think of the retrograde/direct orbits as obtained from a Hopf link in S^3, via the double cover to $\mathbb{R}P^3$. This is the binding of the open book $\mathbb{R}P^3 = \mathbf{OB}(\mathbb{D}^*S^1, \tau^2)$, where τ is the positive Dehn twist along $S^1 \subset \mathbb{D}^*S^1$.

Brouwer's and Frank's theorem. In order to find the direct orbit away from the lunar problem, Birkhoff had in mind finding a disk-like surface of section whose boundary is precisely the retrograde orbit. The direct orbit would then be found via Brouwer's translation theorem: every area preserving homeomorphism of the open disk admits a fixed point. Removing the fixed point, we obtain an area preserving homeomorphism of the open annulus, which, via a theorem of Franks, admits either none or infinitely many periodic points. All this combined, one has: an area preserving homeomorphism of an open disk admits either one or infinitely many periodic points. Note that if the boundary is also an orbit, we obtain 2 or infinitely many. If furthermore we have twist, the Poincaré-Birkohff theorem provides infinitely many orbits. This is a classical heuristic for finding orbits that has survived to this day in several guises. See Figure 11.

Perturbative results. Smoothly, as we have seen, we have $\mathbb{R}P^3 = \mathbf{OB}(\mathbb{D}^*S^1, \tau^2)$, and one would hope that a concrete version of this open book is adapted to the (Moser-regularized) planar dynamics, and that the return map is a Birkhoff twist map. For $c < H(L_1)$ and $\mu \sim 0$ small, one can interpret from this perspective that Poincaré [P12] proved this by perturbing the rotating Kepler problem (when $\mu = 0$), which is an integrable system for which the return map is a twist

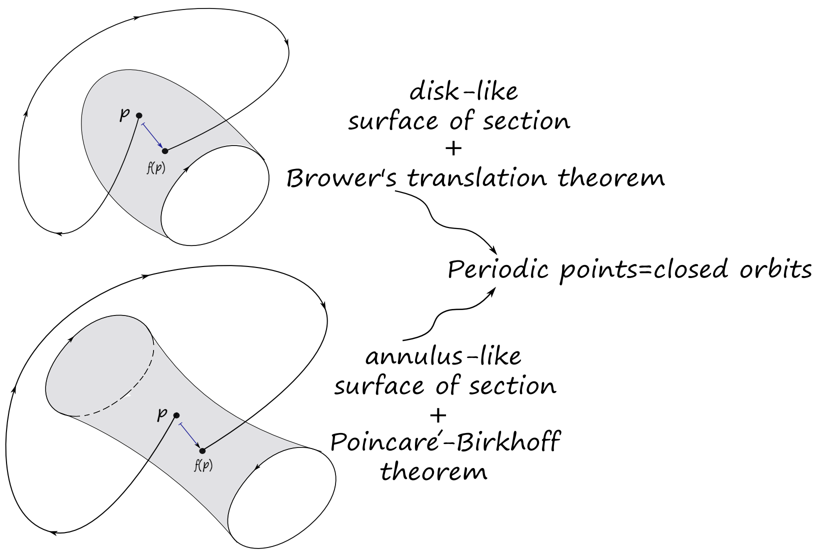

FIGURE 11. Obtaining closed orbits in the planar problem.

map. In the case where $c \ll H(L_1)$ is very negative and $\mu \in (0,1)$ is arbitrary, this was done by Conley [C63] (also perturbatively), who checked the twist condition and used Poincaré-Birkhoff. In [M69], McGehee provides a disk-like global surface of section for the rotating Kepler problem problem for $c < H(L_1)$ and μ arbitrary, and computes the return map.

Non-perturbative results. More generally and *non-perturbatively*, the existence of this adapted open book was obtained in [HSW, Thm. 1.18] for the case where (μ, c) lies in the *convexity range* via holomorphic curve methods due to Hofer-Wysocki-Zehnder [HWZ98] (see also [AFFHvK, AF-FvK]). We will discuss this non-perturbative approach below.

As a final remark for this section, we point out that the advantage of KAM theory (in the perturbative case), when compared to more abstract approaches via general fixed point theorems, is that in favourable situations one can localize periodic (or quasi-periodic) orbits in bounded regions of phase-space, and obtain better qualitative information on these. This is, of course, much more complicated in non-perturbative situations, where rigorous numerics is usually the preferred approach.

References. A nice basic introduction to the classical KAM theorem is e.g. [W08]. Another very nice exposition on the basics behind Mather theory is e.g. [S15]. A beautiful and very detailed account on the three-body problem and Poincaré's work are the notes by Chenciner [Ch15].

6. CONTACT GEOMETRY IN THE RESTRICTED THREE-BODY PROBLEM.

The next result opens up the possibility of using techniques from contact geometry on the CR3BP:

Theorem K ([AFvKP] (planar problem), [CJK] (spatial problem))**.** *If $c < H(L_1)$, the Moser-regularized energy hypersurfaces $\overline{\Sigma}_c^E$ and $\overline{\Sigma}_c^M$ are contact-type. The same holds for $\Sigma_c^{E,M}$, if $c \in (H(L_1), H(L_1)+\epsilon)$*

for sufficiently small $\epsilon > 0$. As contact manifolds, we have

$$\overline{\Sigma}_c^E \cong \overline{\Sigma}_c^M \cong (S^*S^3, \xi_{std}),\ \textit{if } c < H(L_1),$$

and

$$\overline{\Sigma}_c^M \cong (S^*S^3, \xi_{std}) \# (S^*S^3, \xi_{std}),\ \textit{if } c \in (H(L_1), H(L_1) + \epsilon).$$

□

Recall that the above just means that there exists a Liouville vector field which is transverse to the regularized level sets; in fact, this is just the fiber-wise Liouville vector field $q\partial_q$. We will refer as the *low-energy range* to the interval $(-\infty, H(L_1) + \epsilon)$ of energies c for which the above result holds.

Remark 6.1. To this day, it is unknown whether the contact condition is lost for sufficiently high Jacobi constant c, but there is strong evidence that suggests that it indeed does not hold.

Remark 6.2 (Weinstein handles)**.** In the above statement, the connected sum is to be interpreted in the contact category; this amounts to attaching a *Weinstein* 1-handle to the disjoint union of two copies of (S^*S^3, ξ_{std}). Roughly speaking, this means removing two Darboux balls and identifying their boundaries via attaching a 1-handle, which is endowed with the extra structure of a symplectic form which glues well to the symplectization form of the standard contact form at each copy. The result is a *Liouville/Weinstein cobordism* having $(S^*S^3, \xi_{std}) \bigsqcup (S^*S^3, \xi_{std})$ at the negative end, and $(S^*S^3, \xi_{std}) \# (S^*S^3, \xi_{std})$ at the positive one. Note that here the terms positive/negative are relevant: the Liouville vector field is outwards/inwards pointing at the corresponding boundary components, respectively, and so these cobordisms are oriented. This is always the local Morse-theoretical picture for a non-degenerate index 1 critical point of a Hamiltonian (as is the case of L_1). To learn about Weinstein manifolds, see e.g. [CieEli]; this source also provides deep connections between this notion and that of Stein manifolds.

References. For a very detailed and well-exposed overview of contact geometry and holomorphic curves in the planar case of the CR3BP, we refer to Frauenfelder-van Koert [FvK18]. Indeed, the subject of this book is precisely the direction outlined in these lecture notes, but focused on the planar problem, and so the reader is specially encouraged to delve in it.

6.1. **Non-perturbative methods: holomorphic curves.** We now discuss the non-perturbative approach coming from the theory of holomorphic curves.

Hofer-Wysocki-Zehnder. We begin with a definition. A connected compact hypersurface $\Sigma \subset \mathbb{R}^4$ is said to *be strictly convex* if there exists a domain $W \subset \mathbb{R}^4$ and a smooth function $\phi : \mathbb{R}^4 \to \mathbb{R}$ satisfying:

(i) (Regularity) $\Sigma = \{\phi = 0\}$ is a regular level set;
(ii) (Bounded domain) $W = \{z \in \mathbb{R}^4 : \phi(z) \leq 0\}$ is bounded and contains the origin; and
(iii) (Positive-definite Hessian) $\nabla^2\phi_z(h, h) > 0$ for $z \in W$ and for each non-zero tangent vector $h \in T\Sigma$.

In this case, the radial vector field is transverse to Σ, and so Σ is a contact-type 3-sphere, inheriting a contact form α induced by the standard Liouville form in $\mathbb{R}^4$.

Remark 6.3. In the planar restricted three-body problem, the values of energy/mass ratio (c, μ) for which the Levi-Civita regularization is strictly convex is called the *convexity range*.

In [HWZ98], Hofer-Wysocki-Zehnder prove the following:

Theorem L. *[HWZ98] A strictly convex hypersurface $(\Sigma, \alpha) \subset \mathbb{R}^4$ has either 2 or infinitely many periodic orbits.*

The strategy of the proof is finding a disk-like global surface of section, and use the combination Brouwer-Franks mentioned as a heuristics above. The difficulty is precisely finding the section. These are to be thought of as the (holomorphic) pages of a trivial open book on $\Sigma \cong S^3 = \mathbf{OB}(\mathbb{D}^2, \mathbb{1})$, which is adapted to the given Reeb dynamics. The rough idea is as follows.

Consider the symplectization $(M, \omega) = (\mathbb{R} \times \Sigma, d(e^t\alpha))$ of (Σ, α). Its tangent space splits as $TM = \xi \oplus \langle \partial_t, R_\alpha \rangle$. A (cylindrical, α-compatible) almost complex structure is an endomorphism $J \in \mathrm{End}(TM)$ satisfying:

- $J^2 = -\mathbb{1}$ (i.e. J is a "90-degree rotation" at each tangent space);
- $J(\xi) = \xi$, $J(\partial_t) = R_\alpha$;
- J is $\mathbb{R}$-invariant;
- $g = d\alpha(\cdot, J\cdot)$ defines a J-invariant Riemannian metric on ξ.

A *J-holomorphic plane* is then a map

$$u : (\mathbb{C}, i) \to (M, J),$$

intertwining the complex structures, i.e. satisfying the non-linear *Cauchy-Riemann* equation

$$J \circ du = du \circ i.$$

The *Hofer-energy* of such a plane is the quantity

$$\mathbf{E}(u) = \sup_{\varphi \in \mathcal{P}} \int_{\mathbb{C}} u^*\omega_\varphi,$$

where $\mathcal{P} = \{\varphi : \mathbb{R} \to (0,1) : \varphi' \geq 0\}$ is the set of orientation preserving diffeomorphisms between $\mathbb{R}$ and $(0,1)$, and $\omega_\varphi = d(e^{\varphi(t)}\alpha)$ is a symplectic form. The choice of J implies that the integrand is point-wise non-negative and so $\mathbf{E}(u) \geq 0$. A fundamental property is that non-constant finite energy J-holomorphic planes are asymptotic to closed Reeb orbits (originally noted by Hofer in his proof of the Weinstein conjecture for overtwisted contact 3-manifolds [H93]):

Proposition 6.4. *[HWZ98, Thm. 2.2] If $\mathbf{E}(u) < +\infty$ and $u = (a, v) \in \mathbb{R} \times \Sigma$ is non-constant, then $0 < \int v^* d\alpha := T < +\infty$, and there exists a sequence $R_k \to +\infty$ such that $\lim_k u(R_k e^{2\pi i t}) = \gamma(tT)$, for a closed Reeb orbit γ.*

Moreover, under a non-degeneracy condition for γ, the above convergence is exponential and $\lim_R u(Re^{2\pi i t}) = \gamma(tT)$, $\lim_R a(Re^{2\pi i t}) = +\infty$. A further fundamental property is *positivity of intersections*; since M is 4-dimensional, generically two planes intersect at a finite number of points, and if they are holomorphic the intersection numbers are positive. However, there is an an obvious drawback: planes are non-compact and so the classical intersection pairing is not homotopy invariant, since intersections can disappear to infinity. The solution to this issue was provided by Siefring [Sie11], who, using the very explicit asymptotic behaviour of finite energy planes, defined an intersection pairing with all the desired properties. In particular, it is homotopy invariant, takes into consideration interior intersections as well as those "coming from infinity", and two holomorphic planes have vanishing Siefring intersection if and only if their images do not intersect at all. Moreover, in such a case, their projections to Σ do not intersect unless their images coincide. (As the attentive reader might have already noticed, Siefring's work is posterior to the above result; but we will ignore this for the purposes of this rough discussion).

With these preambles, the main idea for the proof of Thm. L is as follows. One assumes the existence of a special Reeb orbit γ, in the sense that is unknotted and linked to every other Reeb orbit (necessary conditions to be the binding of a trivial open book for S^3), non-degenerate, has minimal period, and satisfies $\mu_{CZ}(\gamma) = 3$. Here, we use the *Conley-Zehnder* index μ_{CZ}, which is roughly speaking a winding number associated to paths of symplectic matrices which are suitably non-degenerate, and is used to assign to every Reeb orbit γ an integer $\mu_{CZ}(\gamma)$ (which depends on a trivialization of the tangent bundle along a choice of disk bounded by γ; in the case of S^3, where $\pi_2(S^3) = 0$, this is independent on choices). One then considers the moduli space $\mathcal{M}$ of finite energy J-holomorphic planes asymptotic to this Reeb orbit γ, and having vanishing Siefring self-intersection, modulo the action of $\mathbb{R}$-translation in the image (recall J is $\mathbb{R}$-invariant) and conformal reparametrizations of the domain $\mathbb{C}$. Its expected dimension is $\dim \mathcal{M} = \mu_{CZ}(\gamma) - 2 = 1$, by the Riemann-Roch formula for the Fredholm index. Moreover, the miraculous 4-dimensional phneomenon of automatic transversality shows that $\mathcal{M}$ is a manifold for any cylindrical J. The properties of the Siefring pairing implies that the projections of planes in $\mathcal{M}$ are immersed, do not intersect, and provide a local foliation of Σ. A further step needed in order to get a global foliation is a way to compactify $\mathcal{M}$. This is provided by Gromov's compactification (or the SFT compactification), obtained by adding strata of nodal curves and "holomorphic buildings" with potentially several "floors"; strictly speaking, these a priori are no longer planes. However, the fact that γ is linked to every other orbit can be used to show that no extra strata needs to be added to $\mathcal{M}$, and is in fact *a priori* compact. The result is that $\mathcal{M} \cong S^1$, and projecting plane in $\mathcal{M}$ to Σ provides a global foliation of Σ. The leaves of this foliation are the S^1-family of pages of an open book with binding γ, and are in fact global surfaces of section for the Reeb dynamics.

While the assumption on the existence of γ above might seem far-fetched, it is implied by *dynamical convexity* [HWZ98, Thm. 1.3]. One says that (Σ, α) is dynamically convex if $\mu_{CZ}(\gamma) \geq 3$ for Reeb every orbit γ. This condition is implied by strict convexity [HWZ98, Thm. 3.4]; intuitively, this implies that there is "enough winding" of the linearized Reeb flow along each orbit (and so, at the end of the day when the open book is obtained, this condition applied to the binding γ implies that the arising return map extends to the boundary). The special Reeb orbit is found by first considering the case of an ellipsoid, in which it is explicitly found, then interpolating to the dynamically convex case by considering a symplectic cobordism, and finally using properties of finite energy planes in cobordisms; see Section 4 in [HWZ98].

Conclusion. The main message to take away from this discussion is that the global surfaces of section are the (holomorphic) pages of a trivial open book on $\Sigma \cong S^3 = \mathbf{OB}(\mathbb{D}^2, \mathbb{1})$, which is *a posteriori* adapted to the given Reeb dynamics. The way that this result ties up with the planar CR3BP is via the Levi-Civita regularization; one says that (μ, c) lies in the convexity range whenever the Levi-Civita regularization is stritly convex (cf. Prop. 4.3). The holomorphic open book provided by HWZ, given suitable symmetries, descends to a *rational* open book on the Moser-regularized hypersurface $\mathbb{R}P^3$ (i.e. the pages are disks, but their boundary is doubly covered). Alternatively, [HSW, Thm. 1.18] provides an honest open book with annuli fibers for $\mathbb{R}P^3 = \mathbf{OB}(\mathbb{D}^*S^1, \tau^2)$, adapted to the planar dynamics. This circle of ideas has also been fruitfully exploited in e.g. [H12, H14]; see [HS20] for a very nice survey and references therein, especially for the applications on the planar CR3BP.

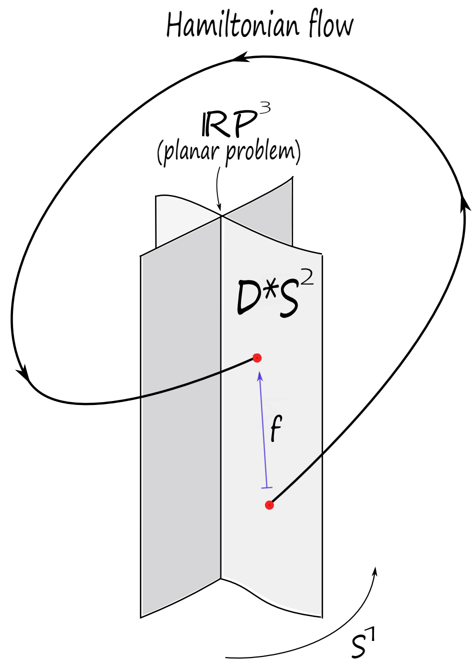

FIGURE 12. The open book for Σ_c, with $c < H(L_1)$, and the first return map f.

7. HOLOMORPHIC CURVE TECHNIQUES ON THE SPATIAL CR3BP.

In this section, we present some (yet unpublished) results of the author, in co-authorship with Otto van Koert. The main direction is to generalize the approach of Poincaré in the planar problem (i.e. Steps (1) and (2) outlined above) to the *spatial* problem.

7.1. **Step (1): Global hypersurfaces of section.** We first state a structural result:

Theorem M (Moreno–van Koert [MvK]). *Denote a connected, bounded component of the regularized, spatial, circular restricted three-body problem for energy level c by Σ_c (for any mass ratio $\mu \in (0,1)$). Let H be the corresponding Hamiltonian, and let L_1 be the first Lagrangian critical point. Then Σ_c is of contact-type and admits a supporting open book decomposition for energies $c < H(L_1)$ that is adapted to the Hamiltonian dynamics of H. Furthermore, there is $\epsilon > 0$ such that the same holds for $c \in (H(L_1), H(L_1)+\epsilon)$. The open books have the following abstract form:*

$$\Sigma_c = \begin{cases} \boldsymbol{OB}(\mathbb{D}^*S^2, \tau^2), & \text{if } c < H(L_1) \\ \boldsymbol{OB}(\mathbb{D}^*S^2 \natural \mathbb{D}^*S^2, \tau_1^2 \circ \tau_2^2), & \text{if } c \in (H(L_1), H(L_1)+\epsilon). \end{cases}$$

Here $\mathbb{D}^*S^2$ is the unit cotangent bundle of the 2-sphere, τ is the positive Dehn-Seidel twist along the Lagrangian zero section $S^2 \subset \mathbb{D}^*S^2$, and $\mathbb{D}^*S^2 \natural \mathbb{D}^*S^2$ denotes the boundary connected sum of two copies of $\mathbb{D}^*S^2$. The monodromy of the second open book is the composition of the square of the positive Dehn-Seidel twists along both zero sections.

See Figure 12 for an abstract representation. We wish to emphasize that Theorem M holds for c in the whole low-energy range. A heuristical reason is the following: while in the planar case

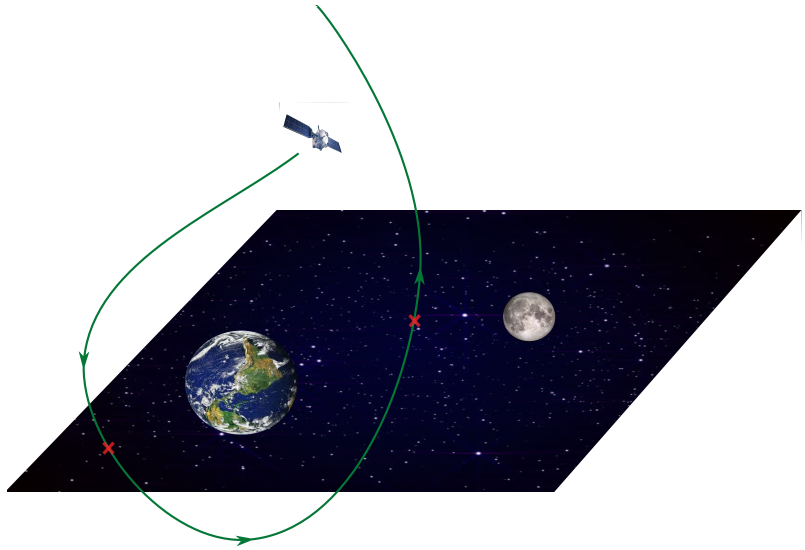

FIGURE 13. Theorem M admits a physical interpretation: away from collisions, the orbits of the negligible mass point intersect the plane containing the primaries transversely. This is intuitively clear from a physical perspective, and translates (after regularization) to the fact that the "pages" $\{q_3 = 0, p_3 > 0\}$, $\{q_3 = 0, p_3 < 0\}$ of the "physical" open book are global hypersurfaces of section outside of the collision locus. Unfortunately this does not extend continuously to the latter, as explained in Figure 14. The binding is the planar problem.

finding the invariant subset is non-trivial (the search for the direct and retrograde orbits indeed has a long history), the invariant subset in the spatial case is immediately obvious; it is the planar problem. The technique of proof does not rely on holomorphic curves, since one can directly write down the open book explicitly.

The above result is motivated by the following observation. We consider a Stark-Zeemaan system satisfying Assumptions (A1) and (A2). In unregularized (or physical) coordinates, we put

$$B_u := \{(\vec{q}, \vec{p}) \in H^{-1}(c) \mid q_3 = p_3 = 0\},$$

the planar problem. Its normal bundle is trivial, and we have the following map to S^1:

$$\pi_p : H^{-1}(c) \setminus B_u \longrightarrow S^1, (\vec{q}, \vec{p}) \longmapsto \frac{q_3 + ip_3}{\|q_3 + ip_3\|}. \tag{7.11}$$

We will refer to this map as the *physical* open book. We consider the angular 1-form

$$\omega_p := d\pi_u := \frac{\Omega_p^u}{p_3^2 + q_3^2},$$

where

$$\Omega_p^u = p_3 dq_3 - q_3 dp_3, \tag{7.12}$$

is the unregularized numerator. We need to see whether $\omega_p(X_H)$ is non-negative, and vanishes only along the planar problem.

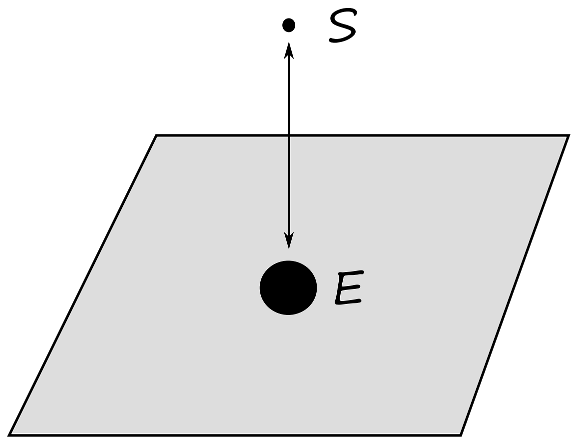

FIGURE 14. There exist (regularized) collision orbits which are periodic and "bounce" vertically over a primary, always staying on the region $q_3 > 0$ (or $q_3 < 0$). This means that the "pages" $\{q_3 = 0, p_3 > 0\}$, $\{q_3 = 0, p_3 < 0\}$ are *not* transverse to the regularized dynamics.

From Equation (4.3), we have

$$\omega_p(X_H) = \frac{p_3^2 + q_3^2 \left(\frac{g}{\|\vec{q}\|^3} + \frac{1}{q_3} \frac{\partial V_1}{\partial q_3}(\vec{q}) \right)}{p_3^2 + q_3^2}. \tag{7.13}$$

Note that Assumption (A2) implies that $\frac{\partial V_1}{\partial q_3}(\vec{q}) = a q_3 + O(q_3^2)$ near $q_3 = 0$, and so $\frac{1}{q_3}\frac{\partial V_1}{\partial q_3}(\vec{q})$ is well-defined at $q_3 = 0$. In order for the above expression to satisfy the required non-negativity condition, we impose the following:

Assumption. (A3) We assume that the function

$$F(\vec{q}) = \frac{g}{\|\vec{q}\|^3} + \frac{1}{q_3} \frac{\partial V_1}{\partial q_3}(\vec{q})$$

is everywhere positive.

Note that it suffices that the second summand be non-negative.

Remark 7.1. In the restricted three-body problem, from Equation (4.7), we obtain

$$\frac{\partial V_1}{\partial q_3}(\vec{q}) = q_3 \frac{1 - \mu}{\|\vec{q} - \vec{e}\|^3},$$

and therefore the corresponding expression in Equation (7.13) is non-negative, vanishing if and only if $p_3 = q_3 = 0$.

The obvious problem of the above computation is that it a priori does not extend to the collision locus, and indeed it cannot (see Figure 14). In fact, one needs to interpolate with the *geodesic* open book described in Section 2.6, which is well-behaved near the collision locus. This creates an interpolation region where fine estimates are needed, and this is the main difficulty in the proof; we refer to [MvK] for the details.

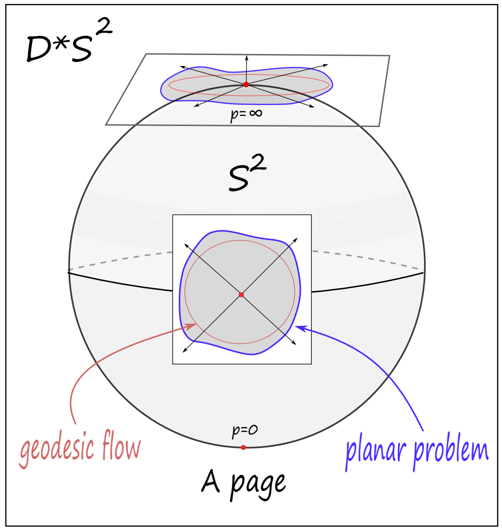

FIGURE 15. A page of the open book as a symplectic filling of the planar problem, viewed as a fiber-wise star-shaped domain in T^*S^2. The geodesic flow corresponds to the unit cotangent bundle.

The return map. For $c < H(L_1)$, and after fixing a page $P = P_1 = \pi^{-1}(1) \cong \mathbb{D}^*S^2$ of the corresponding open book, Theorem M implies the existence of a Poincaré return map $f : \text{int}(\mathbb{D}^*S^2) \to \text{int}(\mathbb{D}^*S^2)$. We say that a symplectomorphism $f : (M, \omega) \to (M, \omega)$ is *Hamiltonian* if $f = \phi_K^1$, where $K : \mathbb{R} \times M \to \mathbb{R}$ is a smooth (time-dependent) Hamiltonian, and ϕ_K^t is the Hamiltonian isotopy it generates. This is defined by $\phi_K^0 = id$, $\frac{d}{dt}\phi_K^t = X_{K_t} \circ \phi_K^t$, and X_{H_t} is the Hamiltonian vector field of H_t defined via $i_{X_{H_t}}\omega = -dH_t$. Here we write $K_t = K(t, \cdot)$.

Theorem N (Moreno–van Koert [MvK]). *For every $\mu \in [0,1]$, $c < H(L_1)$, the associated Poincaré return map extends smoothly to the boundary $\partial\mathbb{D}^*S^2 = \mathbb{R}P^3$, to an exact symplectomorphism*

$$f = f_{c,\mu} : (\mathbb{D}^*S^2, \omega) \to (\mathbb{D}^*S^2, \omega),$$

where $\omega = d\alpha$ (depending on c, μ) is deformation equivalent to the standard symplectic form ω_{std} on $\mathbb{D}^*S^2$. Moreover, f is Hamiltonian, generated by a (not *necessarily autonomous nor rel boundary) Hamiltonian isotopy ϕ_K^t which preserves the boundary.*

The symplectic form ω is the restriction to a given page P of $d\alpha$, where α is the contact form on Σ_c for the spatial problem, whose restriction to the binding α_P is the contact form for the planar problem. Then (P, ω) is a Liouville filling of (B, α_P). The form ω can be symplectically deformed, in the class of Liouville fillings of the fixed contact *structure* on B, to the standard sympletic form by deforming to the Kepler problem (the limit $c \to -\infty$, for which f is the identity). Equivalently, we can think of P as having the standard symplectic form, but non-standard contact boundary (as in Figure 15).

The fact that f is an exact symplectomorphism follows from Prop. 2.16. The fact that f extends to the boundary is non-trivial, and relies on second order estimates near the binding: it suffices to show that the Hamiltonian giving the spatial problem is positive definite on the symplectic normal bundle to the binding. This nondegeneracy condition can be interpreted as a convexity condition that plays the role, in this setup, of the notion of dynamical convexity due to Hofer-Wysocki-Zehnder. Note that if a continuous extension exists, then by continuity it is unique.

The fact that f is Hamiltonian where the isotopy is boundary-preserving follows from:

(1) The monodromy $\phi = \tau^2$ is Hamiltonian, via a boundary-preserving isotopy;
(2) The map $f \circ \phi^{-1}$ is symplectically trivial via a boundary-preserving isotopy, whenever f is a return map arising from a Reeb dynamics on an open book with monodromy ϕ, which also happens to extend to the boundary;
(3) $H^1(P; \mathbb{R}) = 0$, so that every symplectic isotopy is Hamiltonian.

7.2. **Step (2): Fixed-point theory of Hamiltonian twist maps.** The periodic points of τ are either boundary periodic points, which give planar orbits, or interior periodic points which are in 1:1 correspondence with spatial orbits. We are interested in finding *interior* periodic points.

The Hamiltonian twist condition. We propose a generalization of the twist condition introduced by Poincaré, for the Hamiltonian case and for arbitrary Liouville domains. Let $(W, \omega = d\lambda)$ be a $2n$-dimensional Liouville domain, and consider a Hamiltonian symplectomorphism τ. Let $(B, \xi) = (\partial W, \ker \alpha)$ be the contact manifold at the boundary where $\alpha = \lambda|_B$, and R_α the Reeb vector field of α. The Liouville vector field V_λ is defined via $i_{V_\lambda}\omega = \lambda$.

Definition 7.2. (**Hamiltonian twist map**) We say that τ is a *Hamiltonian twist map* (with respect to α), if τ is generated by a *smooth* Hamiltonian $H : \mathbb{R} \times W \to \mathbb{R}$ which satisfies $X_{H_t}|_B = h_t R_\alpha$ for some *positive* and smooth function $h : \mathbb{R} \times B \to \mathbb{R}^+$.

In particular, $H_t|_B \equiv const$ on B, and $\tau(B) \subset B$. We have $h_t = dH_t(V_\lambda)|_B$ is the derivative of H_t in the Liouville direction V_λ along B, which we assume strictly positive. Also, $\tau|_B$ is the time-1 map of a positive reparametrization of the Reeb flow on B. But note that, while the latter condition is only localized at B, the twist condition is of a *global* nature, as it requires global smoothness of the generating Hamiltonian (cf. [MvK, Rk. 1.3]).

Here is a simple example illustrating why the smoothness of the Hamiltonian is relevant for the purposes of fixed points:

Example 7.3 (Integrable twist maps). Let $M = S^n$ for $n \geq 1$ with the round metric, and $H : T^*M \to \mathbb{R}$, $H(q,p) = 2\pi|p|$ (*not* smooth at the zero section); ϕ_H^1 extends to all of $\mathbb{D}^*M$ as the identity. It is a positive reparametrization of the Reeb flow at S^*M, a full turn of the geodesic flow, and all orbits are fixed points with fixed period. If we smoothen H near $|p| = 0$ to $K(q,p) = 2\pi g(|p|)$, with $g(0) = g'(0) = 0$, then $\tau = \phi_K^1 : \mathbb{D}^*M \to \mathbb{D}^*M$, $\tau(q,p) = \phi_H^{2\pi g'(|p|)}(q,p)$, is now a Hamiltonian twist map. If $g'(|p|) = l/k \in \mathbb{Q}$ with l, k coprime, then τ has a simple k-periodic orbit; therefore τ has simple interior orbits of arbitrary large period (cf. [KH95, p. 350], [M86], for the case $M = S^1$).

Remark 7.4. In what follows, we shall appeal to the symplectic homology (or the Floer homology) of a Liouville domain (W, λ), denoted $SH_\bullet(W, \lambda)$. This is a homology theory, introduced originally by Viterbo [V18, V99], which keeps track of both dynamical and topological data; it is, roughly speaking, the homology of a chain complex generated by critical points of a Morse function on the interior of W, as well as by Reeb orbits at the boundary ∂W. These are the 1-periodic orbits of an *admissible* Hamiltonian, i.e. linear at infinity and C^2-small and Morse in the interior. Formally, one needs to take a direct limit over admissible Hamiltonians whose slope increases to infinity, so that we capture orbits at the boundary with all possible periods. The grading in symplectic homology comes from the Conley-Zehnder index (whenever orbits are non-degenerate); for the degenerate case, one can also use the Robbin-Salamon index. The details behind its definition are beyond the scope of this survey; we refer e.g. to [BO09, CO18].

The Hamiltonian twist condition will be used to extend the Hamiltonian to a Hamiltonian that is admissible for computing symplectic homology. The extended Hamiltonian can have additional 1-periodic orbits and these, as well as 1-periodic orbits on the boundary, need be distinguished from the interior periodic points of τ. We impose the following conditions to do so.

Index growth. We consider a suitable index growth condition on the dynamics on the boundary, which is satisfied in the three-body problem whenever the *planar* dynamics is strictly convex. This assumption will allow us to separate boundary and extension orbits from interior ones via the index.

We call a strict contact manifold $(Y, \xi = \ker \alpha)$ *strongly index-definite* if the contact structure $(\xi, d\alpha)$ admits a symplectic trivialization ϵ with the property that

- There are constants $c > 0$ and $d \in \mathbb{R}$ such that for every Reeb chord $\gamma : [0, T] \to Y$ of Reeb action $T = \int_0^T \gamma^*\alpha$ we have
$$|\mu_{RS}(\gamma; \epsilon)| \geq cT + d,$$
where μ_{RS} is the Robbin–Salamon index [RS93].

Index-positivity is defined similarly, where we drop the absolute value. A variation of this notion was explored in Ustilovsky's thesis [U99]. He imposed the additional condition $\pi_1(Y) = 0$. With this extra assumption, the concept of index positivity becomes independent of the choice of trivialization, although the exact constants c and d still depend on the trivialization ϵ. The global trivialization is important when considering extensions of our Hamiltonians, as it allows us to measure the index growth of potential new orbits. The point in the above definition is that the index of boundary orbits grows to infinity under iterations of our return map, and so these do not contribute to symplectic homology.

A general condition for index-positivity to hold, which is also relevant for the restricted three-body problem, is the following:

Lemma 7.5. *Suppose that (Σ, α) is a strictly convex hypersurface in $\mathbb{R}^4$. Then (Σ, α) is strongly index-positive.*

Fixed-point theorems. We propose the following generalization of the Poincaré–Birkhoff theorem:

Theorem O (Moreno–van Koert [MvK]. Generalized Poincaré–Birkhoff theorem)**.** *Suppose that τ is an exact symplectomorphism of a connected Liouville domain (W, λ), and let $\alpha = \lambda|_B$. Assume the following:*

- ***(Hamiltonian twist map)*** *τ is a Hamiltonian twist map, where the generating Hamiltonian is at least C^2;*
- ***(index-definiteness)*** *If $\dim W \geq 4$, then assume $c_1(W)|_{\pi_2(W)} = 0$, and $(\partial W, \alpha)$ is strongly index-definite. In addition, assume all fixed points of τ are isolated;*
- ***(Symplectic homology)*** *$SH_\bullet(W)$ is infinite dimensional.*

Then τ has simple interior periodic points of arbitrarily large (integer) period.

Remark 7.6. Let us discuss some aspects of the theorem:

(1) **(Grading)** We impose the assumptions $c_1(W)|_{\pi_2(W)} = 0$ (i.e. W is symplectic Calabi-Yau) to have a well-defined integer grading on symplectic homology.
(2) **(Surfaces)** If $\dim W = 2$, then the condition that $SH_\bullet(W)$ is infinite dimensional just means that W is not D^2; for D^2 we have $SH_\bullet(D^2) = 0$, and a rotation on D^2 gives an obvious counterexample to the conclusion. In the surface case, the argument simplifies, and one can simply work with homotopy classes of loops rather than the grading on symplectic homology. The Hamiltonian twist condition recovers the classical twist condition for $W = \mathbb{D}^*S^1$, due to orientations.
(3) **(Cotangent bundles)** The symplectic homology of the cotangent bundle of a closed manifold is well-known to be infinite dimensional, due to a result of Viterbo [V18, V99] (see also [AS06]), combined e.g. with a theorem of Gromov [G78, Sec. 1.4]. We have $c_1(T^*M) = 0$ whenever M is orientable. As for the existence of a global trivialization of the contact structure $(\xi, d\lambda_{can})$, we note the following:
   - if Σ is an oriented surface, then $S^*\Sigma$ admits such a global symplectic trivialization;
   - if M^3 is an orientable 3-manifold, then S^*M^3 admits such a global symplectic trivialization.
   - In addition, symplectic trivializations of the contact structure on (S^*S^2, λ_{can}) are unique up to homotopy.
(4) **(Fixed points)** If fixed points are non-isolated, then we vacuously obtain infinitely many of them, although we cannot conclude that their periods are unbounded; "generically", one expects finitely many fixed points.
(5) **(Long orbits)** If W is a global hypersurface of section for some Reeb dynamics, with return map τ, interior periodic points with long (integer) period for τ translates into spatial Reeb orbits with long (real) period.
(6) **(Katok examples)** There are well-known examples due to Katok [K73] of Finsler metrics on spheres with only finitely many simple geodesics, which are arbitrarily close to the round metric; they admit global hypersurfaces of section with Hamiltonian return maps, for which the index-definiteness and the condition on symplectic homology hold. It follows that the return map does not satisfy the twist condition for any choice of Hamiltonians.
(7) **(Spatial restricted three-body problem)** From the above discussion and [MvK], we gather: the only standing obstruction for applying the above result to the spatial restricted three-body problem, in case where the planar problem is strictly convex, is the Hamiltonian twist condition. Here, note that symplectic homology is invariant under deformations of Liouville domains; see e.g. [BR] for a paper with detailed proofs. This would give a proof of existence of *spatial* long orbits in the spirit of Conley [C63], which could in principle be collision orbits. Since the geodesic flow on S^2 arises as a limit case (i.e. the Kepler problem), it should be clear from the discussion on Katok examples that this is a subtle condition. In [MvK], we have computed a generating Hamiltonian for the integrable case of the rotating Kepler problem; it does *not* satisfy the twist condition in the spatial case (in the planar case, a Hamiltonian twist map was essentially found by Poincaré). This does not mean a priori that there is not *another* generating Hamiltonian which does, but this seems rather unlikely.

As a particular case of Thm. O, we state the above result for star-shaped domains in cotangent bundles, as of independent interest (cf. [H11]):

Theorem P (Moreno–van Koert [MvK]). *Suppose that W is a fiber-wise star-shaped domain in the Liouville manifold (T^*M, λ_{can}), where M is simply connected, orientable and closed, and assume that $\tau : W \to W$ is a Hamiltonian twist map. If the Reeb flow on ∂W is strongly index-positive, and if all fixed points of τ are isolated, then τ has simple interior periodic points of arbitrarily large period.*

The above also holds for $M = S^2$, as explained in Remark 7.6 (2). The difference with [H11] is that in this setup we conclude that periodic points are interior, to the expense of imposing index-positivity and the twist condition.

7.3. **Alternative approach: dynamics on moduli spaces.** An alternative approach to that of an abstract fixed-point theorem, is the following (also abstract) construction. We start by recalling that the page $\mathbb{D}^*S^2 = \mathbf{LF}(\mathbb{D}^*S^1, \tau_P^2)$ of the open book of Thm. M has a Lefschetz fibration with genus zero fibers over the 2-disk, with monodromy the Dehn twist τ_P (P here is for "planar", to differentiate from the monodromy τ used for the spatial case; recall Figure 8). The main geometric observation for what follows is: the leaf space $\mathcal{M}$ of such fibers (i.e. the moduli space parametrizing them) is a copy of S^3. Indeed, each page $\mathbb{D}^*S^2$ of the open book $S^2 \times S^3 = \mathbf{OB}(\mathbb{D}^*S^2, \tau^2)$ is a 2-disk worth of fibers; we moreover have an S^1-family of such pages, all of them sharing the boundary $\mathbb{R}P^3$ (the binding), and such that their Lefschetz fibration all induce the S^1-family of pages of the open book $\mathbb{R}P^3 = \mathbf{OB}(\mathbb{D}^*S^1, \tau_P^2)$. It follows that the leaf space carries the trivial open book $\mathcal{M} = \mathbf{OB}(\mathbb{D}^2, \mathbb{1}) \cong S^3$, whose disk-like page corresponds to the base of the page in $S^2 \times S^3$, and whose binding $\mathcal{M}_B$ is the S^1-family of pages for $\mathbb{R}P^3$. See Figure 16.

Rotating Kepler problem. In [MvK, App. A], we discuss the completely integrable limit case of the rotating Kepler problem, where $\mu = 0$ and so there is only one primary. The return map can be studied completely explicitly. Geometrically, this map may be understood via the following proposition (recall Figure 8):

Proposition 7.7 ([MvK], Integrable case). *In the rotating Kepler problem, the return map f preserves the annuli fibers of the standard Lefschetz fibration $\mathbb{D}^*S^2 = \mathbf{LF}(\mathbb{D}^*S^1, \tau_P^2)$, where it acts as a classical integrable twist map on regular fibers, and fixes the two (unique) nodal singularities on the singular fibers.*

The two fixed points are the north and south poles of the zero section S^2, and correspond to the two periodic collision orbits bouncing on the primary (one for each of the half-planes $q_3 > 0$, $q_3 < 0$).

The abstract case. We now consider an abstract situation where the previous argument also holds. Consider a *concrete* open book decomposition $\pi : M \backslash B \to S^1$ on a contact 5-manifold $(M, \xi_M) = \mathbf{OB}(P, \phi)$. We assume that P (abstractly) admits the structure of a 4-dimensional Lefschetz fibration over $\mathbb{D}^2$ whose fibers are surfaces of genus zero and perhaps several boundary components. We abstractly write $P = \mathbf{LF}(F, \phi_F)$, where ϕ_F is the monodromy of the Lefschetz fibration on P (as we have discussed, necessarily a product of positive Dehn twists on the genus zero surface F).

Following [Acu], we will refer to the open book on M as an *iterated planar* (IP) open book decomposition, and the contact manifold M as iterated planar. As observed in [AEO, Lemma 4.1], a contact 5–manifold is iterated planar if and only if it admits an open book decomposition supporting the contact structure, whose binding is planar (i.e. admits a 3-dimensional supporting open book whose pages have genus zero). In fact, we have $B = \mathbf{OB}(F, \phi_F)$.

We wish to adapt the underlying planar structure to a *given* Reeb dynamics on M (and hence the need to work with concrete open books, rather than the abstract version). We then assume that the

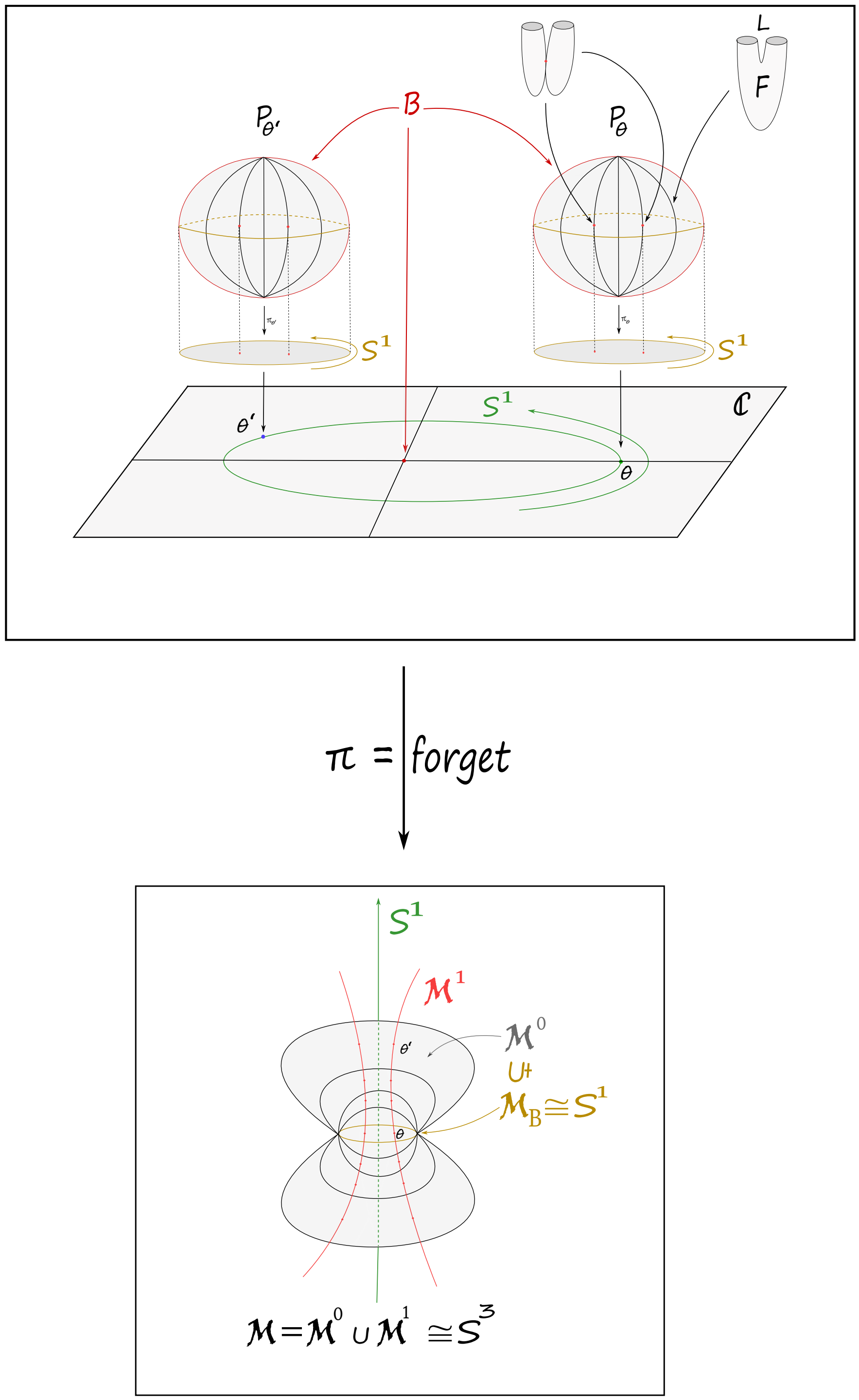

FIGURE 16. The moduli space $\mathcal{M} \cong S^3$ has two strata: the open strata $\mathcal{M}^0$ consisting of regular fibers, and the nodal strata $\mathcal{M}^1$ consisting of singular fibers.

concrete open book on M is adapted to the Reeb dynamics of a *fixed* contact form α_M, i.e. α_M is a Giroux form for the open book (whose dynamics we wish to study). In particular, $\omega_\theta := d\alpha_M|_{P_\theta}$ is a symplectic form on P_θ for each $\theta \in S^1$. Therefore $(P_\theta, \omega_\theta)$ is a Liouville filling of the binding $(B, \xi_B = \ker \alpha_B)$, where $\alpha_B = \alpha_M|_B$, for each θ. We will further assume that we have a *concrete* planar open book on the 3-manifold $B = \mathbf{OB}(F, \phi_F)$, which is adapted to the Reeb dynamics of α_B and where ϕ_F is a product of positive Dehn twists in the genus zero surface F. We will denote $L = \partial F$, which is a link in B (the binding of the open book for B, and Reeb orbits for α_B). Given the above situation, we will say that the Giroux form α_M is an *IP Giroux form*.

This is precisely the situation in the SCR3BP whenever the planar dynamics is strictly convex/dynamically convex, as follows from [HSW, Thm. 1.18], combined with Thm. M above. We now state the general construction:

Theorem Q ([M20], IP foliation)**.** *There is a foliation $\overline{\mathcal{M}}$ of $M \backslash L$, consisting of immersed $d\alpha_M$-holomorphic curves whose boundary is L. Away from B, its elements are arranged as fibers of Lefschetz fibrations $\pi_\theta : P_\theta \to \mathbb{D}^2_\theta$, $\theta \in S^1$, all of which induce the same fixed concrete open book at B. The π_θ are all generic, i.e. each fiber contains at most a single critical point. We have $\mathcal{M} \cong S^3$, and it is endowed with the trivial open book whose θ-page is identified with $\mathbb{D}^2_\theta$, and its binding is $\mathcal{M}_B \cong S^1$, the family of pages of the open book at B.*

The point here is that the above result is in principle non-perturbative; it applies whenever there is an adapted open book at B. It should be thought of as an S^1-parametric version of Wendl's result (Thm. J above). We can further endow the moduli space with extra structure:

Theorem R ([M20], contact and symplectic structures on moduli)**.** *The moduli space $\mathcal{M}$ carries a natural contact structure $\xi_{\mathcal{M}}$ which is supported by the trivial open book on S^3 (and hence it is isotopic to the standard contact structure ξ_{std} by Giroux correspondence). Moreover, the symplectization form on $\mathbb{R} \times M$ associated to any Giroux form α_M on M induces a tautological symplectic form on $\mathcal{M}$ by leaf-wise integration, which is naturally the symplectization of a contact form $\alpha_{\mathcal{M}}$ for $\xi_{\mathcal{M}}$, whose Reeb flow is adapted to the trivial open book on $\mathcal{M}$.*

The holomorphic shadow. We define the *holomorphic shadow* of the Reeb dynamics of α_M on M to be the Reeb dynamics of the associated contact form $\alpha_{\mathcal{M}}$ on S^3, provided by Theorem R. The flow of $\alpha_{\mathcal{M}}$ can be viewed as a flow $\phi_t^{M;\mathcal{M}}$ on $M \backslash L$ which leaves the holomorphic foliation $\mathcal{M}$ invariant (i.e. it maps holomorphic curves to holomorphic curves). It is the "best approximation" of the Reeb flow of α_M with this property, as its generating vector field is obtained by projecting the original Reeb vector field to the tangent space of $\mathcal{M}$, via a suitable L^2-orthogonal projection. It may also be viewed as a Reeb flow $\phi_t^{S^3;\mathcal{M}}$ on S^3, related to the one on M via a semi-conjugation

$$\begin{array}{ccc} M\backslash L & \xrightarrow{\phi_t^{M;\mathcal{M}}} & M\backslash L \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_*^q} \\ S^3 & \xrightarrow{\phi_t^{S^3;\mathcal{M}}} & S^3 \end{array}$$

where π is the projection to the leaf-space $\mathcal{M} \cong S^3$. We will now focus on the global properties of the correspondence $\alpha_M \mapsto \alpha_{\mathcal{M}}$.

For F a genus zero surface, let $\mathbf{Reeb}(F, \phi_F)$ denote the collection of contact forms whose flow is adapted to a (fixed) concrete planar open book $\pi_B : B \backslash L \to S^1$ on a given 3-manifold B, of abstract form $B = \mathbf{OB}(F, \phi_F)$. Iteratively, we define $\mathbf{Reeb}(\mathbf{LF}(F, \phi_F), \phi)$ to be the collection of contact forms with flow adapted to a (fixed) concrete IP open book $\pi_M : M \backslash B \to S^1$ on a 5-manifold M, of abstract form $M = \mathbf{OB}(\mathbf{LF}(F, \phi_F), \phi)$, whose restriction to the binding $B = \mathbf{OB}(F, \phi_F)$ belongs to $\mathbf{Reeb}(F, \phi_F)$. We call elements in $\mathbf{Reeb}(\mathbf{LF}(F, \phi_F), \phi)$ *IP contact forms*, or *IP Giroux forms*.

We then have a map

$$\mathbf{HS}_J : \mathbf{Reeb}(\mathbf{LF}(F, \phi_F), \phi) \to \mathbf{Reeb}(\mathbb{D}^2, \mathbb{1}),$$

given by taking the holomorphic shadow with respect to an auxiliary almost complex structure J associated to α_M. We have the normalization $\mathbf{HS}_J(\alpha_M) = \alpha_{std}$, where the flow of α_{std} is the Hopf flow (we refer to $\mathbf{HS}_J^{-1}(\alpha_{std})$ as the *integrable fiber*).

Theorem S (Reeb flow lifting theorem)**.** $\mathbf{HS}_J$ *is surjective.*

In other words, for a given J, we may lift any Reeb flow on S^3 adapted to the trivial open book, as the holomorphic shadow of the Reeb flow of an IP Giroux form adapted to *any* choice of concrete IP contact 5-fold. The map $\mathbf{HS}_J$ is clearly not in general injective, as it forgets dynamical information in the fibers. While the above lifting procedure is not precisely an extension of the flow, morally the above theorem says that Reeb dynamics on an IP contact 5-fold is at least as complex as Reeb dynamics on the standard contact 3-sphere. Somewhat related, we point out that higher-dimensional Reeb flows encode the complexity of all flows on arbitrary compact manifolds (i.e. they are *universal*) [CMPP].

Dynamical Applications. We wish to apply the above results to the SCR3BP. We first introduce the following general notion. Consider an IP 5-fold M with an IP Reeb dynamics, endowed with an IP holomorphic foliation $\mathcal{M}$ as in Theorem Q. Fix a page P in the IP open book of M, and consider the associated Poincaré return map $f : \text{int}(P) \to \text{int}(P)$. A (spatial) point $x \in \text{int}(P)$ is said to be *leaf-wise* (or *fiber-wise*) k*-recurrent* with respect to $\mathcal{M}$ if $f^k(x) \in \mathcal{M}_x$, where $\mathcal{M}_x$ is the leaf of $\mathcal{M}$ containing x, and $k \geq 1$. This means that $f^k(\text{int}(\mathcal{M}_x)) \cap \text{int}(\mathcal{M}_x) \neq \emptyset$. This is, roughly speaking, a symplectic version of the notion of *leaf-wise intersection* introduced by Moser [M78] for the case of the isotropic foliation of a coisotropic submanifold.

In the integrable case of the rotating Kepler problem, where the mass ratio $\mu = 0$, the holomorphic foliation provided by Theorem Q can be obtained by restriction to S^*S^3 of the standard Lefschetz fibration on T^*S^3 (when T^*S^3 is viewed as a quadric in $\mathbb{C}^4$; see the discussion on T^*S^2 above). Denote this "integrable" holomorphic foliation on S^*S^3 by $\mathcal{M}_{int}$. Since the return map for $\mu = 0$ preserves fibers, every point is leaf-wise 1-recurrent with respect to $\mathcal{M}_{int}$. If the mass ratio is sufficiently small, then the leaves of $\mathcal{F}_{int}$ will still be symplectic with respect to $d\alpha$, where α is the corresponding perturbed contact form on the unit cotangent bundle S^*S^3.

We have the following perturbative result:

Theorem T. *In the SCR3BP, for any choice of page P in the open book of Thm. M, for any fixed choice of $k \geq 1$, for sufficiently small μ (depending on k), for energy c below the first critical value $H(L_1(\mu))$, along the bounded components of the Hill region, and for every $l \leq k$, there exist infinitely many points in int(P) which are leaf-wise l-recurrent with respect to $\mathcal{M}_{int}$.*

Remark 7.8. The same conclusion holds for arbitrary $\mu \in [0, 1]$, but sufficiently negative $c \ll 0$ (depending on μ and k).

In fact, the conclusion of the Theorem T holds whenever the relevant return map is sufficiently close to a return map which preserves the leaves of the holomorphic foliation of Theorem Q (i.e.

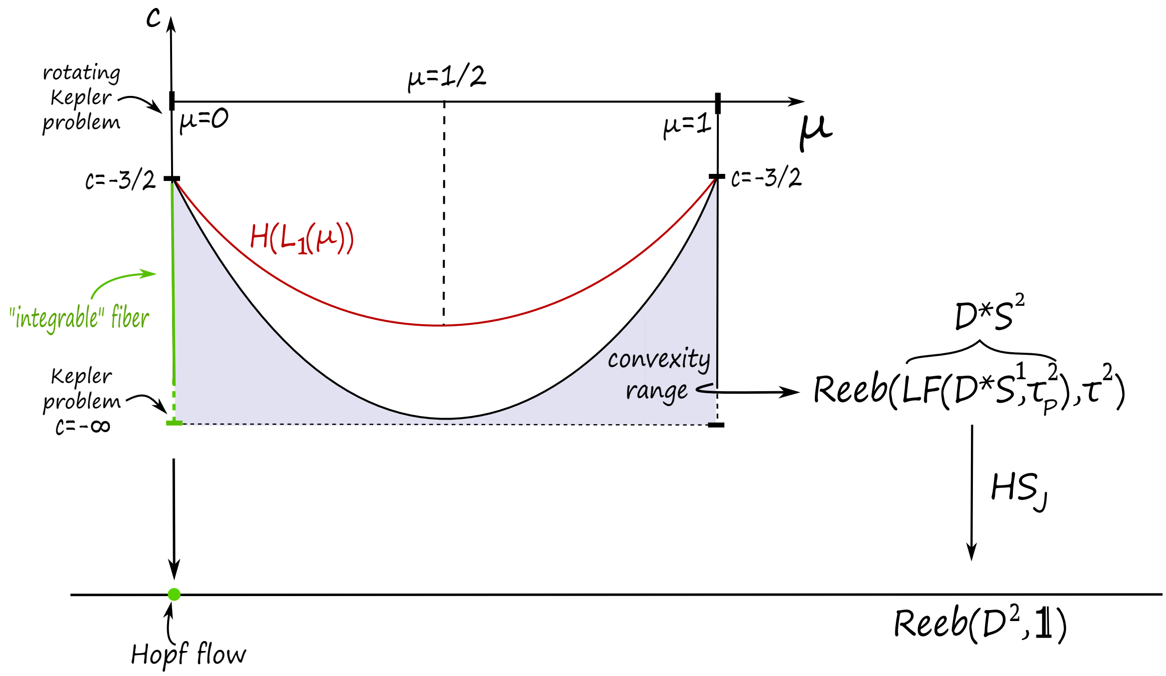

FIGURE 17. An abstract sketch of the convexity range in the SCR3BP (shaded), for which the holomorphic shadow is well-defined. We should disclaim that the above is not a plot; the convexity range is not yet fully understood, although it contains (perhaps strictly) a region which qualitatively looks like the above, cf. [AFFHvK, AFFvK].

which coincides with its holomorphic shadow on M). It may then be interpreted as a symplectic version of the main theorem in [M78], for two-dimensional symplectic leaves. The advantage of considering the integrable foliation (in terms of applications) is that it can be written down explicitly in complex coordinates.

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