Johann Heinrich Lambert and his theorem about Keplerian orbits

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J.H. Lambert (1728-1777) is a "repeatedly forgotten genius". He had many fundamental contributions in several domains of science as optics, astronomy, geometry or analysis.

Lambert gave a theorem about the time Δt needed to go from a point A to a point B with a given total energy H on a Keplerian orbit around a point O: Δt is unchanged if the Keplerian orbit is changed continuously in such a way that H, the distance $d(\mathrm{A},\mathrm{B})$, the sum $d(\mathrm{O},\mathrm{A})+d(\mathrm{O},\mathrm{B})$ remain unchanged. This statement is simple and without exception but unintuitive and surprising. There are surprising difficulties to check even a particular case.

We announce with Zhao Lei a new result which is related to several of Lambert's contributions. Our new result is that this theorem is still true on a space of constant curvature. We can see at least four relations.

1) It is a direct generalization of his Theorem of 1761. The proof of this Theorem is complicated if one does not follow our method. The proof of the generalization is more difficult due to the complexity of the expression of the time in the Kepler problem on constant curvature spaces. But we completely avoid this computational complication.

2) Lambert is among the famous authors who improved our understanding of Euclid's fifth postulate. He wrote in 1766 the essay "Theorie der Parallellinien" which was published posthumously in 1786. The possibility of removing the fifth axiom and still keep geometrical properties became a standard subject of study, which was then extended to dynamics by Paul Serret in 1859. We make a step forward in this direction.

3) Lambert is also one of the founders of abstract projective geometry. He wrote in 1759 "La perspective affranchie de l'embaras du Plan géometral". In 1890, Appell showed that the dynamics of a particle possesses an invariance by projective transformations. We use this invariance to deduce our extension through a central projection of the flat case.

4) The key formula for such a deduction was given by Hamilton in 1834 at the beginning of the works where he founded Hamiltonian dynamics. He got such formula by a variational reasoning inspired by optics. Lambert was also among the founders of advanced geometrical optics. He wrote in 1759 a book "Les proprietés remarquables de la route de la lumière".